

Course notes for Real Analysis

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They are updated online at:

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Lecture 1: March 31

Banach Spaces

Definition 1.1. A Banach space is a complete normed vector space.

1. $L^p(\mu)$ (X, \mathcal{M}, μ) complete σ -finite measure space, for $p \in [1, \infty]$. Recall membership in L^p spaces; proof of completeness using absolute continuity.
2. X a topological space, $B(X)$ and $BC(X)$ are both Banach space in the uniform norm:
 $\|f\|_\infty = \sup_{x \in X} |f(x)|$
3. N_α . Hölder continuous functions, $0 < \alpha \leq 1$.

$$\Lambda^\alpha(\mathbb{R}^d) = \{f: \mathbb{R}^d \rightarrow \mathbb{R}: \|f\|_\alpha < \infty\}$$

and

$$\|f\|_\alpha = \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

you use Arzela-Ascoli to get candidate function, use Cauchy to get stuff to work.

4. Building normed vector spaces from existing ones:

$$(X, \|\cdot\|_X), (Y, \|\cdot\|_Y) \mapsto (X \times Y, \|\cdot\|)$$

where

$$\|(x, y)\| = \begin{cases} \max\{\|x\|_X, \|y\|_Y\} \\ \|x\|_X + \|y\|_Y \\ \sqrt{\|x\|_X^2 + \|y\|_Y^2} \end{cases}$$

They are all equivalent (it is an easy exercise to see that they all bound one another).

5. **Quotient Spaces:**

X vector space, \mathcal{M} linear subspace. For any $x, y \in X$, say $x \sim y$ if and only if $x - y \in \mathcal{M}$. Look at a norm X/\mathcal{M} which is the set of equivalence classes and is a vector space, under the operations $(x + \mathcal{M}) + (y + \mathcal{M}) = (x + y) + \mathcal{M}$, and $\lambda(x + \mathcal{M}) = \lambda x + \mathcal{M}$.

If X has a norm $\|\cdot\|$ and \mathcal{M} is **closed**, then X/\mathcal{M} inherits the norm $\|x + \mathcal{M}\| = \inf_{y \in \mathcal{M}} \|x + y\|$ called the quotient norm. When $X = \mathcal{H}$ is a Hilbert space, then there is a particularly nice characterization in terms of distance from x to y_x , which is the closest point to x along the subspace \mathcal{M} .

Definition. X, Y normed vector spaces.

1. A linear map $T: X \rightarrow Y$ is bounded if there is an $M > 0$ for which $\|Tx\| \leq M\|x\|$.
2. A linear map is continuous if for all $\epsilon > 0$, there is a $\delta > 0$ such that for $f, g \in X$ with $\|f - g\| < \delta$, we have $\|Tf - Tg\| < \epsilon$.

Comment: It looks like uniform, and something else looks like Lipschitz. But we have the following proposition:

Let $T: X \rightarrow Y$ be a linear map. The following are equivalent:

1. T is continuous
2. T is continuous at 0
3. T is bounded

Claim 1.2. T is linear, T is bounded, and $\|T - T_n\| \rightarrow 0$.

Proof. Since $\{T_n\}$ is Cauchy, it is bounded. Let $M = \sup_n \|T_n\| < \infty$. Then

$$\begin{aligned} \|Tx\| &= \left\| \lim_{n \rightarrow \infty} T_n x \right\| \\ &\leq \limsup_{n \rightarrow \infty} \|T_n\| \|x\| \\ &\leq M \|x\|. \end{aligned}$$

So we get $\{T_n\}$ bounded.

Recall

$$\|T - T_n\| = \sup\{\|(T - T_n)(x)\| : \|x\| = 1\};$$

for each n , then for all $m \in \mathbb{N}$ there is an x_m^n such that $\|x_m^n\| = 1$ and

$$\|T - T_n\| \leq \|Tx_m^n - T_n x_m^n\| + \frac{1}{2^m}.$$

Also $\lim_{j \rightarrow \infty} T_j x_m^n = Tx_m^n$. Then there is a $j_0 \geq n$ such that for all $j \geq j_0$, we have $\|T_j x_m^n - Tx_m^n\| < 2^{-m}$. Then

$$\begin{aligned} \|T - T_n\| &\leq \|Tx_m^n - T_n x_m^n\| + 2^{-m} \\ &\leq \|Tx_m^n - T_j x_m^n\| + \|T_j x_m^n - T_n x_m^n\| + 2^{-m} \\ &\leq 2^{m-1} + \|T_n x_m^n - T_j x_m^n\| \\ &\leq 2^{m-1} + \|T_n - T_j\|. \end{aligned}$$

Then for $j \geq j_0 \geq n$, there is an $N \in \mathbb{N}$ with $j \geq n \geq N$. Since $\{T_l\}$ Cauchy, $\|T_n - T_j\| < 2^{m-1}$ and we get $\|T - T_n\| < 2^{m-2}$. \square

Lecture 2: April 2

Dual of L^p

In today's lecture, (X, \mathcal{M}, μ) is a complete σ -finite measure space (not absolutely necessary for all cases). Suppose $p \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Try to identify the dual of $L^p(\mu)$; start with an example.

Given $g \in L^q(\mu)$, define

$$Tf = \int fg \, d\mu \quad \text{linear functions}$$

By Hölder, $|Tf| \leq \|g\|_q \|f\|_p$; thus by definition of the operator norm, $\|T\| \leq \|g\|_q$. The goal of today's class is to prove that for $p < \infty$, there is nobody else; these are all the functionals.

Theorem 2.1. *Suppose $p \in [1, \infty)$, and p, q are dual. Then $(L^p(\mu))^* = L^q(\mu)$ (* means space of bounded linear functionals).*

This is in the following sense: if $\ell \in (L^p(\mu))^*$, then there is a unique $g \in L^q(\mu)$ such that $\ell(f) = \int fg \, d\mu$ and $\|\ell\| = \|g\|_q$ for all $f \in L^p(\mu)$. Also note that the theorem is not true when $p = \infty$; later on we'll see what the dual is.

Lemma 2.2. *S*

1. If $g \in L^q(\mu)$, then

$$\|g\|_q = \sup_{f \in L^p, \|f\|_q \leq 1} \left| \int fg \, d\mu \right|.$$

(this computation is useful in PDE)

2. Suppose that g is measurable and integrable on sets of finite measure and

$$M = \sup \left\{ \int fg \, d\mu : f \in L^p(\mu), f \text{ is simple}, \|f\|_p \leq 1 \right\} < \infty$$

Then $g \in L^q(\mu)$ and $\|g\|_q = M$.

Proof. Radon-Nikodym states that if μ is σ -finite, ν is a signed σ -finite measure, and if $|\nu| \ll \mu$ then $\exists!$ g μ -measurable such that $d\nu = g d\mu$.

The first step is if $\mu < \infty$; then we'll do σ -finite. So assume μ is finite. Then if $E \in \mathcal{M}$, we have $\mu(E) < \infty$ hence $\chi_E \in L^p(\mu)$. Then define $\nu(E) = \ell(\chi_E)$; want to show it is a measure.

First $\nu(\emptyset) = 0$. Now let $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{M}$ such that $E_i \cap E_j = \emptyset$. Let

$$E = \cup E_i = \cup_{i=1}^n E_i \cup E_n^*, \quad E_n^* = \cup^{\infty} E_i.$$

Then $\cap E_n^* = \emptyset$ and $\mu(E_n^*) < \infty$. Thus $\lim_{n \rightarrow \infty} \mu(E_n^*) = 0$. Since ℓ is linear, we have

$$\begin{aligned} \ell(\chi_E) &= \ell(\chi_{\cup_{i=1}^n E_i}) + \ell(\chi_{E_n^*}) \\ &= \sum_{i=1}^n \ell(\chi_{E_i}) + \ell(\chi_{E_n^*}) \\ &= \sum_{i=1}^n \nu(E_i) + \nu(E_n^*). \end{aligned}$$

We want to use a continuity of measure type of argument. We need to prove that $\nu(E_n^*) \rightarrow 0$. Observe

$$|\nu(E)| = |\ell(\chi_E)| \leq \|\ell\| \|\chi_E\|_p = \|\ell\| (\mu E)^{1/p},$$

so that $|\nu(E_n^*)| \leq \|\ell\| (\mu E_n^*)^{1/p}$ (the first sign that $p < \infty$ matters).

Thus we have proven that $\nu(E) = \ell(\chi_E)$ is a signed measure, $\nu < \infty$, and (by Hölder again) $|\nu| \ll \mu$. So by Radon-Nikodym, there is a unique measurable g such that $d\nu = g d\mu$ and $g \in L^1(\mu)$. Moreover $\ell(\chi_E) = \nu(E) = \int_E g d\mu$. If f is simple (defined a little different than first quarter but it's fine), then by linearity $\ell(f) = \int f g d\mu$. Therefore $|\ell(f)| \leq \|\ell\| \|f\|_p$.

$$\begin{aligned} \sup_{\|f\|_p \leq 1, f \text{ simple}} |\ell(f)| &= \sup \left| \int f g d\mu \right| \\ &\leq \|\ell\| < \infty. \end{aligned}$$

Thus by (2) of Lemma (as yet unproven), $g \in L^q(\mu)$ and $\|g\|_q \leq \|\ell\|$. Hence if μ is finite and we assume our Lemma, we are done for simple functions. By density of simple functions, we are actually done.

Now suppose μ is σ -finite, so $X = \cup_{i \in \mathbb{N}} X_i$ where $\mu(X_i) < \infty$. Arrange for $X_n \subset X_{n+1}$; consider $\mu \perp X_n$ (the restriction - definition is first line of manipulation). Then

$$L^p(X_n, \mu) = \{f \in L^p(\mu) : f = 0 \text{ on } X_n^c\},$$

and since $\mu \perp X_n(A) = \mu(A \cap X_n) \leq \mu X_n < \infty$, there is a $g_n \in L^q(\mu \perp X_n)$ such that for $f \in L^p(\mu \perp X_n)$ we have $\ell(f) = \int g_n f d\mu$.

Claim 2.3. *If $n < m$, then $g_n = g_m$ μ -everywhere on X_n .*

Proof. If $f \in L^p(\mu \perp X_n)$ then $f \in L^p(\mu \perp X_m)$. Observe that $\int f g_n d\mu = \int f g_m d\mu$. Hence $g_m|_{X_n} = g_n$ a.e., by uniqueness in Radon-Nikodym. \square

Let $g = g_n$ if $x \in X_n$. Now $g_n \in L^q(\mu \perp X_n)$, by MCT; since $0 \leq |g_n| \leq \dots \leq |g|$. Now $|g_n| \rightarrow |g|$ μ -a.e. by MCT. Thus

$$\|g\|_q = \lim_{n \rightarrow \infty} \|g_n\|_q \leq \|\ell\|.$$

To show that last line, we need to assume $q < \infty$ (so that we can take q -powers of both sides). If $q = \infty$, we have

$$\|g\|_\infty = \sup_n \|g_n\|_{L^\infty(X_n)} \leq \ell.$$

Now if $f \in L^p(\mu)$ then $f\chi_{X_n} \rightarrow f$ μ -a.e., and by LDCT we get $|f\chi_{X_n} - f| \rightarrow 0$ in $L^p(\mu)$ (use a $(2f)^p$ argument). Again using MCT, we get

$$\begin{aligned} \int fg \, d\mu &= \lim_{n \rightarrow \infty} \int_{X_n} fg_n \, d\mu \\ &= \lim_{n \rightarrow \infty} \ell(f\chi_{X_n}) \\ &= \ell(f) \end{aligned}$$

□

Lecture 3: April 4

Theorem 3.1. $p \in [1, \infty)$, $(L^p(\mu))^* = L^q(\mu)$.

Lemma 3.2. $1 \leq p, q \leq \infty$ and p, q dual; everything is over the reals, not complex (but it is still true) (X, \mathcal{M}, μ) a σ -finite complete measure space.

1. If $g \in L^q(\mu)$ then $\|g\|_q = \sup\{\int fg \, d\mu\} = \Lambda$ (taken over $\|f\|_p \leq 1$).
2. If g is measurable and integrable on sets of finite measure and $M = \sup \int fg \, d\mu$ (supremum taken over f simple and $\|f\|_p \leq 1$) satisfies $M < \infty$, then $g \in L^q(\mu)$ and $M = \|g\|_q$.

Proof. Proof of 1: If $g = 0$ we're done; so assume $g \neq 0$, i.e. there is a set of positive measure where g is non-zero; $\|g\|_q \neq 0$. By Hölder, $\Lambda \leq \|g\|_q$. So we need to choose f to be the equality case.

Case 1: $q = 1$. Let $f(x) = \text{sgn } g(x)$, which is measurable (prove it). Then $\|f\|_\infty = 1$. Hence

$$\begin{aligned} \int fg \, d\mu &= \int \text{sgn } g(x) \cdot g(x) \, d\mu \\ &= \int |g| \, d\mu \\ &= \|g\|_1 \\ &\leq \Lambda \end{aligned}$$

and we're done.

Case 2: $1 < p, q < \infty$. Choose

$$f(x) = |g(x)|^{q-1} \frac{\text{sgn } g(x)}{\|g\|_q^{q-1}}$$

Then $|f|^p = |g|^q / \|g\|_q^q$ so $\|f\|_p = 1$. Also

$$\begin{aligned} \int fg \, d\mu &= \int \frac{|g|^{q-1} \text{sgn } g \cdot g \, d\mu}{\|g\|_q^{q-1}} \\ &= \frac{\|g\|_q^q}{\|g\|_q^{q-1}} \\ &= \|g\|_q \end{aligned}$$

and we're done. Notice that up until now, we haven't used σ -finite; it will come up in the last case.

Case 3: If $q = \infty$, Let $\epsilon > 0$ and E a set with $\mu(E) \in (0, \infty)$ such that $|g| > \|g\|_\infty - \epsilon$ on E . We use σ -finiteness to ensure that E can be chosen to have finite measure. Take $f = \chi_E \operatorname{sgn} g / \mu E$, show that $f \in L^1(\mu)$ and $\|f\|_1 = 1$. Then

$$\begin{aligned} \int fg \, d\mu &= \int_E \frac{|g|}{\mu E} \, d\mu \\ &\geq \|g\|_\infty - \epsilon. \end{aligned}$$

Now $\Lambda = \sup_{\|f\|_1 \leq 1} \int fg \, d\mu \geq \|g\|_\infty - \epsilon$, so letting $\epsilon \rightarrow 0$ we obtain $\Lambda \geq \|g\|_\infty$.

Proof of 2: Since g is measurable, there are simple functions g_n such that $|g_n| \nearrow |g|$ and $g_n \rightarrow g$ almost everywhere. By σ -finiteness, we can write $X_n \nearrow X$ with $\mu X_n < \infty$. Choose $g_n = g_n \chi_{X_n}$, so $g_n \in L^q$.

Let's assume $1 < p, q < \infty$. Construct $f_n = \frac{|g_n|^{q-1} \operatorname{sgn} g}{\|g_n\|_q^q}$; then $f_n \in L^p$. Then

$$\begin{aligned} M &\geq \int f_n g \, d\mu \\ &= \int \frac{|g_n|^{q-1}}{\|g_n\|_q^{q-1}} \\ &\geq \frac{\|g_n\|_q^q}{\|g_n\|_q^{q-1}} \\ &= \|g_n\|_q. \end{aligned}$$

Now by MCT, $\|g\|_q = \lim_{n \rightarrow \infty} \|g_n\|_q \leq M$ and we're done.

If $p = 1$, let $A_\epsilon = \{x: |g(x)| > M + \epsilon\}$. Now show that $\mu A_\epsilon = 0$ for all $\epsilon > 0$, whereupon $\|g\|_\infty \leq M$. Finish the proof (it goes by contradiction). If it was not the case, then $\mu A_\epsilon > 0$, so take $B \subset A_\epsilon$ with $\mu(B) \in (0, \infty)$. Follow the same proof and something happens yielding a contradiction.

If $p = \infty$, then $f_n = \operatorname{sgn} g(x) \chi_{X_n}$. Just integrate it again g and see what happens. \square

Lecture 4: April 7

Definition 4.1. Let X be a real vector space. A sublinear functional $p: X \rightarrow \mathbb{R}$ is a map that satisfies:

1. $p(x + y) \leq p(x) + p(y)$
2. $p(\lambda x) = \lambda p(x) \quad \lambda \geq 0$

Take $p = \|\cdot\|$ for a normed vector space.

Theorem 4.2 (Hahn-Banach). Let X be a real vector space, p a sublinear functional, let \mathcal{M} be a linear subspace of X , let $f: \mathcal{M} \rightarrow \mathbb{R}$ be a linear functional such that $f(z) \leq p(z)$ for $z \in \mathcal{M}$. Then there exists a linear functional $F: X \rightarrow \mathbb{R}$ such that $F|_{\mathcal{M}} = f$ and $F(z) \leq p(z)$ for $z \in X$.

As an application, we show that $(L^\infty(\mu))^* \not\cong L^1(\mu)$.

Take $X = [0, 1]$, $\mu = L^1$, $C(X) \subset L^\infty(\mu)$. Then $T: C(X) \rightarrow \mathbb{R}; f \mapsto f(0)$ satisfies $|f(0)| = |Tf| \leq \|f\|_\infty$ (bounded by sublinear functional). Then by Hahn Banach, we can extend to $F: L^\infty(\mu) \rightarrow \mathbb{R}$ (so that $F|_{C(X)} = T$). Note that $|F(g)| \leq \|g\|_\infty$.

F is $L^\infty(\mu)^*$. Let $f_n = (1 - nx)\chi_{[0, 1/n]}$. Then $F(f_n) = 1$. If F was integration against $g \in L^1$, there would be a problem. Indeed, $f_n g \rightarrow 0$ almost everywhere. Also $|f_n g| \leq |g|$, so by dominated convergence $\int f_n g \, dx \rightarrow 0$. So we've constructed an element of $L^\infty(\mu)^*$ that does not correspond to anything in L^1 .

Hahn-Banach.

Lemma 4.3 (Step 1). If $x \in X \setminus \mathcal{M}$, then f can be extended to g on $\mathcal{M} + \mathbb{R}x$, such that $g \leq p$.

Lemma 4.4 (Step 2). We are going to consider the class $\mathcal{F} = \{g: \mathcal{N} \rightarrow \mathbb{R}: g|_{\mathcal{M}} = f, g \leq p\}$ of all possible extensions of f (where \mathcal{N} is a subspace containing \mathcal{M}).

Step 1. Start by taking $y_1, y_2 \in \mathcal{M}$, x as in the statement. Then

$$f(y_1) + f(y_2) \leq p(y_1 + y_2) \leq p(y_1 - x) + p(y_2 + x).$$

Hence $f(y_1) - p(y_1 - x) \leq p(y_2 + x) - f(y_2)$. Therefore

$$\sup_{y \in \mathcal{M}} \{f(y) - p(y - x)\} \leq \inf \{p(y + x) - f(y)\} = \alpha.$$

Now take $g(y + \lambda x) = f(y) + \lambda \alpha$.

We consider any $y + \lambda x \in \mathcal{M} + \mathbb{R}x$. If $y \in \mathcal{M}$, then $g(y) = f(y) \leq p(y)$ (the $\lambda = 0$ case). If $\lambda > 0$, then from choice of α

$$\begin{aligned} g(y + \lambda x) &= f(y) + \lambda \alpha \\ &= \lambda f(y/\lambda) + \alpha \\ &\leq \lambda p(y/\lambda + x) \\ &= p(y + \lambda x). \end{aligned}$$

If $\lambda < 0$, then

$$\begin{aligned} g(y + \lambda x) &= -\lambda f(-y/x) - \alpha \\ &\leq -\lambda f(-y/\lambda) - f(-y/\lambda) + p(-y/\lambda - x) \\ &\leq p(y + \lambda x). \end{aligned}$$

□

Step 2. $g_1, g_2 \in \mathcal{F}$

Define $g_1 < g_2$ if $\mathcal{N}_1 \subset \mathcal{N}_2$ and $g_2|_{\mathcal{N}_1} = g_1$. By the Hausdorff Maximum Principle (i.e., axiom of choice) there exists a maximally ordered chain $\{g_\alpha\}$. Define $F: \cup_\alpha \mathcal{N}_\alpha \rightarrow \mathbb{R}$ by setting $F(x) = g_{\alpha(x)}(x)$ if $x \in \mathcal{N}_\alpha$. It is clear that $F \leq p$.

1. $\cup_\alpha \mathcal{N}_\alpha$ is a subspace
2. F is well-defined
3. F is a linear functional
4. $F \leq p$ for $x \in \cup \mathcal{N}_\alpha$
5. $F|_{\mathcal{M}} = f$
6. $\cup \mathcal{N}_\alpha = X$.

F is a maximal extension. Indeed, it follows from maximality of the g_α ; for any other maximal extension G , we have $G = g_\alpha$ for some α . Now if $x \in X \setminus \cup \mathcal{N}_\alpha$ then Step 1 would yield a proper extension; so we're done. □

Theorem 4.5. *Let X be a normed vector space.*

1. *If \mathcal{M} is a closed subspace of X and $x \in X \setminus \mathcal{M}$, then there is $f \in X^*$ such that $f|_{\mathcal{M}} = 0$, $\|f\| = 1$ and $f(x) = \inf_{y \in \mathcal{M}} \|y - x\|$.*
2. *If $x \in X \setminus \{0\}$, there is $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*
3. *Bounded linear functionals separate points.*
4. *For $x \in X$, define $\hat{x}: X^* \rightarrow \mathbb{R}$ by $\hat{x}(f) = f(x)$. The map $\hat{\cdot}$ is an isometry. Thus we can embed any normed space in a natural Banach space.*

□

Lecture 5: April 9

Consequences of Hahn-Banach

X is a normed vector space.

1. \mathcal{M} a closed subspace of X , $x \in X \setminus \mathcal{M}$. Then there is an $f \in X^*$ such that $\|f\| = 1$, $f|_{\mathcal{M}} = 0$ and $f(x) = \inf_{y \in \mathcal{M}} \|x - y\|$.

When X is a Hilbert space and \mathcal{M} is a closed subspace, we have already seen something similar. Specifically, there is a nearest point $x_0 \in \mathcal{M}$ where the infimum of the distance is attained. Moreover, we get a result about orthogonality: for all $z \in \mathcal{M}$, we have $\langle x - x_0, z \rangle = 0$. For $x \notin \mathcal{M}$, we have $f(y) = \langle y, \frac{x - x_0}{\|x - x_0\|} \rangle$ is a bounded linear functional. It is clear that $\|f\| \leq 1$, and since $f(x - x_0) = \|x - x_0\|$ we actually have $\|f\| = 1$.

Thus (1) gives us a way to make some statements as if we had an inner product, even though we don't.

Proof of (1). Consider $f: \mathcal{M} + \mathbb{R}x \rightarrow \mathbb{R}$. We set $f(y + \lambda x) = \lambda \delta$, for $y \in \mathcal{M}$. We want to bound it by a sublinear functional. If $\lambda = 0$ we're done, so assume $\lambda \neq 0$. We start with $|f(y + \lambda x)| = |\lambda \delta|$, and want to make some manipulation to get a bound of $\|y + \lambda x\|$. Specifically, using the infimum from the definition of δ we obtain $\lambda \delta \leq |\lambda| \|x + \frac{y}{\lambda}\| = \|y + \lambda x\|$. Now by Hahn-Banach, there is an $F: X \rightarrow \mathbb{R}$ such that $F|_{\mathcal{M} + \mathbb{R}x} = f$. Now $F(x) = \delta$ and $F|_{\mathcal{M}} = 0$. We haven't used yet that \mathcal{M} is closed. It gives us a sequence of points y_ϵ such that $\|x - y_\epsilon\|$ is converging to δ . We can make some argument with a subsequence to get y_ϵ converging somehow.

For $\epsilon > 0$, there is $y_\epsilon \in \mathcal{M}$ such that $\delta \leq \|x - y_\epsilon\| \leq \delta(1 + \epsilon)$. Now $F(x - y_\epsilon) = \delta$, and

$$|F(x - y_\epsilon)| = \delta \geq \frac{\|x - y_\epsilon\|}{1 + \epsilon}$$

□

2. $x \in X$, $x \neq 0$. Then there is $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\| = 1$ (just take $\mathcal{M} = \{0\}$).
3. X^* separates points in X ; i.e., $x \neq y$ means there is an $f \in X^*$ such that $f(x) \neq f(y)$.

Proof. Let $z = x - y \neq 0$. Then choose $f \in X^*$ such that $\|f\| = 1$. $f(x - y) = \|x - y\| > 0$; thus $f(x) \neq f(y)$. □

4. $x \in X$; $\hat{x}: X^* \rightarrow \mathbb{R}$, and $\hat{x}(f) = f(x)$. Show that $\hat{\cdot}: X \rightarrow (X^*)^*$ is an isometry.

Proof. First $|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\|$; thus $\|\hat{x}\| \leq \|x\|$. So we only need to show the other direction. Start by assuming $x \neq 0$. Pick $f \in X^*$ with $\|f\| = 1$ such that $f(x) = \|x\|$. Then $\|\hat{x}\| \geq \|x\|$. The result follows. \square

What this means is that we can always put a complete normed space into a bigger space.

Definition 5.1. A Banach space X is reflexive if $(X^*)^* = X$.

An example is $L^p(\mu)$, when $p \in (1, \infty)$ (follows from $L^p(\mu)^* = L^q$). First we have $(L^1(\mu)) \not\subset (L^1(\mu))^{**}$, thus $L^1(\mu)$ is not reflexive. We don't even know the dual of L^∞ , so we can't talk about the same question when $p = \infty$.

Baire Category Theorem

Definition 5.2. Let X be a topological space. Then $E \subset X$ is **nowhere dense** if $\text{int } \bar{E} = \emptyset$. Equivalently, \bar{E}^c is dense.

For example, the Cantor set. These sets are "swiss cheese"; they have holes everywhere. We will see lots of important applications to make precise statements.

Lecture 6: April 11

Existence of Closest Points

Let X be a Banach space, \mathcal{M} a closed subspace, and consider $x \in X \setminus \mathcal{M}$. We say that x **has a closest point in \mathcal{M}** if there is a $y_x \in \mathcal{M}$ such that

$$\|x - y_x\| = \inf_{y \in \mathcal{M}} \|x - y\| = \delta \quad (*)$$

Today we will learn about spaces that always admit closest points.

Definition 6.1. A Banach space X is **uniformly convex** when for every $\epsilon > 0$ and any $x, y \in X$ with $\|x\| = \|y\| = 1$, if $\|x - y\| > \epsilon$ then $\left\|\frac{x+y}{2}\right\| < 1 - \delta$ and $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

In the Hilbert space case, we used the parallelogram law to prove that a sequence of y_n 's approaching the infimum was Cauchy. It gives a relationship between $\|x - y\|$, $\left\|\frac{x+y}{2}\right\|$ and the norms $\|x\|, \|y\|$.

Claim 6.2. If X is uniformly convex, then $(*)$ holds. In particular, by the homework this holds for every L^p with $p \in (1, \infty)$.

Proof sketch of claim. Fix $x \notin \mathcal{M}$, and pick a sequence $y_k \in \mathcal{M}$ such that $\delta \leq \|x - y_k\| \leq \delta(1 + 2^{-k})$. Note that if $\{y_k\}$ is Cauchy, then since X is complete $y_k \rightarrow y$. Since \mathcal{M} is closed, $y \in \mathcal{M}$.

Let $z_k = \frac{x - y_k}{\|x - y_k\|}$. We show that $\{z_k\}$ is Cauchy if and only if y_k is Cauchy. Indeed,

$$z_k - z_j = \frac{y_j - y_k}{\|x - y_k\|} + (x - y_j) \frac{\|x - y_j\| - \|x - y_k\|}{\|x - y_j\| \|x - y_k\|}$$

Heuristically, the right term is of size $\delta \cdot (\delta 2^{-k}) / \delta^2 = 2^{-k}$.

Assume $\{z_k\}$ is not Cauchy. Then there is an $\epsilon_0 > 0$ such that for all j , there is a $k_j \geq j$ such that $\|z_k - z_j\| \geq \epsilon_0$. By uniform convexity, $\left\|\frac{z_k + z_j}{2}\right\| < 1 - \delta_0$. We want to contradict minimality of δ . But

$$\begin{aligned} \delta \frac{z_k + z_j}{2} &= \frac{x\delta}{2} \left(\frac{1}{\|x - y_k\|} + \frac{1}{\|x - y_j\|} \right) - \frac{\delta}{2} \left(\frac{y_k}{\|x - y_k\|} + \frac{y_j}{\|x - y_j\|} \right) \\ &= \frac{1}{2} \left(\frac{\delta}{\|x - y_k\|} + \frac{\delta}{\|x - y_j\|} \right) x - \frac{\delta}{2} \left(\frac{y_k}{\|x - y_k\|} + \frac{y_j}{\|x - y_j\|} \right) \\ &= M \end{aligned}$$

Note from before that $\|M\| \leq \delta(1 - \delta_0)$. The first term tends to x and the second lies in \mathcal{M} , so we will end up contradicting minimality of δ .

Now

$$\|x - \omega_{jk}\| \leq \|M\| + \|x\| \left\| 1 - \frac{1}{2} \left(\frac{\delta}{\|y_k - x\|} + \frac{\delta}{\|y_j - x\|} \right) \right\| \leq \delta(1 - \delta_0) + \delta\delta_0/2$$

and the result follows. \square

So we have exhibited a family of spaces in which closest points exist. How can we construct an example of a space without minimizers? So we prove a consequence of $(*)$ that's easier to verify, and then we'll construct a counterexample by making a space that doesn't satisfy the consequence.

Let X be a Banach space such that $(*)$ holds for every closed subspace \mathcal{M} and $x \notin \mathcal{M}$. Start with a bounded linear functional f such that $f(x) = 1$ and call $\mathcal{M} = \ker f$. Assume $y \in \mathcal{M}$ and $\|x - y\| = d(x, \mathcal{M})$. Set $z = x - y$. Then $f(z) = f(x) - f(y) = f(x) = 1$. Now $d(z, \mathcal{M}) = d(x, \mathcal{M}) = \|x - y\| = \|z\|$.

Now apply Hahn-Banach. There is a $g \in X^*$ such that $g|_{\mathcal{M}} = 0$ and $\|g\| = 1$. For any $u \in X$, write $f(u) = f(f(u)x)$. Thus $u - f(u)x \in \ker f = \mathcal{M}$, so we have the decomposition $X = \mathcal{M} + \mathbb{R}x$. We can make $g(z) = \|z\|$, so $g = \|z\|f$ and thus $\|g\| = \|z\|\|f\|$. Therefore $\|f\| = \frac{1}{\|z\|}$ and $f(z) = \|f\|\|z\|$.

So if $(*)$ holds, then for all $f \in X^*$ with $f \neq 0$, there is a $z \neq 0$ such that $f(z) = \|f\|\|z\|$. So if we can arrange a space X and $f \in X^*$ such that $|f(z)| < \|f\|\|z\|$ for all $z \neq 0$, it will follow that X lacks minimizers.

Take $X = L^1(\mathbb{R})$, and $f(h) = \int_{\mathbb{R}} h\phi(t) dt$. Here $\phi(t) = \chi(t)_{[-1,1]^c} e^{-1/t^2} + \frac{1}{e}\chi_{[-1,1]}$. Then $|f(z)| < \|f\|\|z\|_1$ and we get a contradiction.

Using more functional analysis, the best result is that minimizers exist if and only if X is reflexive.

Lecture 7: April 14

Baire Category

Definition 7.1. Let X be a topological space. Then $E \subset X$ is nowhere dense if $\text{Int } \bar{E} = \emptyset$ or \bar{E}^c is dense.

Theorem 7.2 (Baire Category). *Let X be a complete metric space. Then*

1. *If $\{U_n\}_{n \in \mathbb{N}}$ is a collection of open dense sets in X , then $\cap U_n$ is dense.*
2. *X is not a countable union of nowhere dense sets.*

Proof. Assuming (1), prove (2). Assume $X = \cup E_n$ of nowhere dense sets. Then also $X = \cup \bar{E}_n$. Taking complements, we get that $\cap \bar{E}_n^c$ is dense yet empty. This is a contradiction, so we're done.

Now prove (2). Let W be an open set, and $\{U_n\}$ a collection of dense open sets in X . We will construct a Cauchy sequence.

Since U_1 is dense, U_1 meets W . Now $U_1 \cap W$ is non-empty and open. Take $x_1 \in U_1 \cap W$. Since it's open, there is a positive $r_1 < \frac{1}{2}$ such that $B(x_1, r_1) \subset \overline{B(x_1, r_1)} \subset U_1 \cap W$.

Since U_2 is open and dense, we can get $x_2 \in B(x_1, r_1) \cap U_2$. Again it's open, so take a positive $r_2 < 2^{-2}$ such that $\overline{B(x_2, r_2)} \subset B(x_1, r_1) \cap U_2$.

Continuing inductively, we construct a sequence $\{x_j\}$ such that $x_{j+1} \in B(x_j, r_j) \cap U_j$, $r_j < 2^{-j}$, and

$$\begin{aligned} \overline{B(x_{j+1}, r_{j+1})} &\subset B(x_j, r_j) \cap U_j \\ &\subset B(x_1, r_1) \cap U_2 \cap U_3 \cap \dots \cap U_j \\ \Rightarrow \overline{B(x_j, r_j)} &\subset W \cap \left(\bigcap_{n=1}^j U_n \right). \end{aligned}$$

Claim 7.3. $\{x_j\}$ is Cauchy

Proof. Take $n, m \geq l$. Then $x_n, x_m \in B(x_l, r_l)$. It follows that $d(x_n, x_m) \leq 2r_l < 2^{1-l}$. □

Since X is complete, the sequence $\{x_n\}$ converges to some x . It follows that $x \in \overline{B(x_j, r_j)}$ for all j , so $x \in W \cap \left(\bigcap_{n=1}^j U_n \right)$. Thus $\bigcap_{n=1}^j U_n$ meets every open set W , and is therefore dense. □

Some silly terminology that is frequently forgotten.

Definition 7.4. X is a topological space.

$E \subset X$ is of first category (or **meager**) if E is a countable union of nowhere dense sets.

If E is not of first category, it is of second category.

The complement of a meager set is said to be **residual** or **generic**.

Don't confuse these with terms from other areas of mathematics!

Corollary 7.5. *In a complete metric space, a generic set is dense (when B is a ball, \bar{B} is complete).*

Theorem 7.6 (Uniform boundedness principle). *Let X be a Banach space, Y a normed vector space, and $A \subset \mathcal{L}(X, Y)$. Suppose that for all $x \in X$, $\sup_{T \in A} \|Tx\| < \infty$. Then $\sup_{T \in A} \|T\| < \infty$.*

This has useful corollaries, for instance Problem 2 from the final in Math 525.

Consider $\{f_n\} \in L^2(\mu)$ with $\sup \|f_n\|_2 < \infty$. Then there is a subsequence $\{f_{n_k}\}$ such that

$$\int f_{n_k} g \, d\mu \rightarrow \int fg \, d\mu, \quad \forall g \in L^2(\mu).$$

Claim 7.7 (A converse statement). *Let $\{g_n\} \in L^2(\mu)$ be a weakly convergent sequence. Then $\sup \|g_n\|_2 < \infty$.*

Proof. Since $\{\int g_n h \, d\mu\}$ converges for h fixed, we have $\sup \|\int g_n h \, d\mu\| < \infty$. Take $A = \{T_n\}$, where $T_n(h) = \int g_n h \, d\mu$. Then $T_n \in \mathcal{L}(L^2(\mu), \mathbb{R})$ where we think of $L^2(\mu)$ as a Banach space now.

$\sup_{T_n \in A} \|T_n(h)\| < \infty$ for all $h \in L^2(\mu)$. Using uniform boundedness followed by Riesz Representation Theorem, $\sup \|T_n\| = \sup \|g_n\|_2 < \infty$ and the result follows. \square

Uniform boundedness principle. Let $E_i = \{x \in X: \sup_{T \in A} \|Tx\| < i\}$. We can also write

$$E_i = \bigcap_{T \in A} \{x \in X: \|Tx\| \leq i\}.$$

Now $X = \cup E_i$. By Baire Category, there is some n such that E_n is not nowhere dense. Since T is continuous and $\|\cdot\|$ is continuous, it follows that E_n is an intersection of closed sets and is therefore closed.

Since $E_n = \overline{E_n}$ and E_n is not nowhere dense, it follows that $\text{Int } E_n \neq \emptyset$. Thus there is an $x_0 \in E_n$ and $r_0 > 0$ such that $\overline{B(x_0, r_0)} \subset B(x_0, 2r_0) \subset E_n$.

Take an $x \in X$ such that $\|x\| \leq r_0$. Then $x + x_0 \in \overline{B(x_0, r_0)} \subset E_n$, so

$$\begin{aligned} \|Tx\| &\leq \|T(x + x_0)\| + \|T(-x_0)\| \\ &\leq 2n. \end{aligned}$$

Thus $\|Tx\| \leq 2n$. Take $\overline{B(0, r_0)} \subset E_{2n}$. Then for all $y \in X \setminus 0$, $r_0 \frac{y}{\|y\|} \in \overline{B(0, r_0)}$. Thus

$$T\left(r_0 \frac{y}{\|y\|}\right) \leq 2n$$

so $\|T\| \leq \frac{2n}{r_0}$. □

Definition 7.8. Let X, Y be topological spaces. Then “ $f: X \rightarrow Y$ is open” means that $f(U)$ is open whenever U is open.

Note. If X, Y are normed vector spaces and $f: X \rightarrow Y$ is linear, to show that f is open it suffices to check that $f(B(0, 1))$ contains some ball $B(0, r)$.

Proof. If f is open, then since $f(0) = 0$ the criterion clearly holds.

Now suppose $U \subset X$ is open. Then take $y \in f(U)$. There is some $x \in U$ such that $y = f(x)$. Then there is a $\delta > 0$ such that $B(x, \delta) \subset U$. But $B(x, \delta) = x + \delta B(0, 1)$. Hence $f(x) + \delta f(B(0, 1)) \subset f(U)$. By hypothesis, there is an $r > 0$ such that $B(0, r) \subset f(B(0, 1))$. Thus

$$B(y, \delta r) \subset f(U)$$

and the result follows. □

Lecture 8: April 16

More Open Mapping Theorem

Theorem 8.1 (Open Mapping Theorem). *Let X, Y be Banach spaces. If $T \in \mathcal{L}(X, Y)$ is surjective, then it is open.*

Recall the corollary from last time:

Corollary 8.2. *If $T \in \mathcal{L}(X, Y)$ is a bijection, then T is an isomorphism.*

This amounts to showing that T^{-1} is continuous.

Open Mapping. From our remark last time, it suffices to show that there is a $\delta > 0$ such that $B(0, \delta) \subset TB(0, 1)$. Write

$$X = \bigcup_{n=1}^{\infty} B_n, \quad B_n = B(0, n) \subset X.$$

Since T is surjective,

$$Y = \bigcup_{n=1}^{\infty} TB_n = \bigcup_{n=1}^{\infty} nTB_1.$$

As Y is complete, by Baire Category there is an n such that nTB_1 is not nowhere dense. Since $\overline{nTB_1} = n\overline{TB_1}$, there is a $y_0 \in \text{Int } \overline{TB_1}$. Hence there is an $r > 0$ such that $B(y_0, 4r) \subset \overline{TB_1}$.

Claim 8.3. $B(0, r) \subset \overline{TB_1}$

Proof. Let $y_1 \in B(y_0, 2r) \cap TB_1 \neq \emptyset$, and take $y_1 = Tx_1$ for $x_1 \in B_1$. Then $B(y_1, 2r) \subset B(y_0, 4r) \subset \overline{TB_1}$. Let $y \in B(0, 2r)$. Then $y = y + y_1 - Tx_1 \in -Tx_1 + \overline{TB_1}$.

Recall $x_1 \in B_1$. Thus $-x_1 + B_1 \subset B_2$. If $y \in B(0, 2r)$ then $y \in \overline{TB_2}$, so $\frac{y}{2} \in \overline{TB_1}$. \square

Remark. For $y \in Y$ with $\|y\| < r2^{-m}$, we have $2^m y \in B(0, r)$. Thus $2^m y \in \overline{TB_1}$. Hence $y \in \overline{2^{-m}TB_1}$.

Claim 8.4. $B(0, \frac{r}{2}) \subset TB_1$

Proof. For $y \in B(0, \frac{r}{2})$ we need $z \in B_1$ such that $y = Tz$. Since $B(0, \frac{r}{2}) \subset \overline{TB_{1/2}}$, we have

$$B(y, \frac{r}{4}) \cap TB_{1/2} \neq \emptyset.$$

Thus there is a $z_1 \in B_{1/2}$ such that $Tz_1 \in B(y, \frac{r}{4})$. Consequently

$$y - Tz_1 \in B(0, \frac{r}{4}) \subset \overline{TB_{1/4}}.$$

Next we get $z_2 \in B_{1/4}$ such that $y - Tz_1 - Tz_2 \in B(0, \frac{r}{8})$. Continuing inductively, we obtain $z_n \in B_{2^{-n}}$ such that

$$y - \sum_{i=1}^n Tz_i \in B\left(0, \frac{r}{2^{n+1}}\right).$$

Note that $\sum_{n=1}^{\infty} \|z_n\| \leq \sum 2^{-n} = 1$, so by completeness of X we have $\sum_{n=1}^{\infty} z_n = z$.

We claim that $y = Tz$. To see it, use continuity of T . Then $Tz = \lim T(\sum z_j) = \lim \sum Tz_j$. Hence $\|y - \sum Tz_i\| < r2^{-n-1}$ so $y = Tz$. We used completeness of Y to apply the Baire Category theorem; injectivity only needed in the corollary. \square

\square

Closed Graph Theorem

Definition 8.5. 1. Let X, Y be normed vector spaces, and T is linear. Then

$$\text{graph } T = \{(x, y) \in X \times Y : y = Tx\}$$

2. T is said to be closed if $\text{graph } T$ is a closed subspaces of $X \times Y$ (note that $\text{graph } T$ is linear).

Remark. If $T \in \mathcal{L}(X, Y)$, then $\text{graph } T$ is closed.

Proof. Assume $(x_n, y_n) \in \text{graph } T$ and $(x_n, y_n) \rightarrow (x, y)$. We need to show that $(x, y) \in \text{graph } T$. Saying $(x_n, y_n) \rightarrow (x, y)$ is the same as saying $x_n \rightarrow x$ and $Tx_n \rightarrow y$. By continuity of T , $Tx_n \rightarrow Tx$. Thus $Tx_n \rightarrow y$. \square

Theorem 8.6 (Closed Graph Theorem). *Let X, Y be Banach and $T: X \rightarrow Y$ linear and closed. Then T is continuous.*

In general, to show that T is continuous it suffices to show that if $x_n \rightarrow x$, then Tx_n converges and its limit is Tx . Using the closed graph theorem, it is enough to show that if $(x_n, Tx_n) \rightarrow (x, y)$, then $y = Tx$.

Proof. Consider the projection $\pi_1, \pi_2: X \times Y \rightarrow X, Y$. These are bounded linear maps. Moreover, $\tilde{\pi}_1 = \pi_1|_{\text{graph } T} \in \mathcal{L}(\text{graph } T, X)$ and likewise for $\tilde{\pi}_2$. Now $\text{graph } T$ is Banach, since it's a closed subset of the complete metric space $X \times Y$. Since $\tilde{\pi}_1$ is surjective, by the open mapping theorem it is open.

If $\pi_1(x, Tx) = \pi_1(z, Tz)$ then $x = z$; hence $\tilde{\pi}_1$ is a bijection. Consequently $\tilde{\pi}_1$ is an isomorphism. Then $\tilde{\pi}_1^{-1}: X \rightarrow \text{graph } T$ is continuous. However, $Tx = \pi_2 \circ \tilde{\pi}_1^{-1}(x)$; thus T is continuous as desired. \square

Lecture 9: April 18

Definition 9.1. Let X be a Banach space.

1. A sequence $\{x_n\} \subset X$ is **weakly convergent** if

$$\lim_{n \rightarrow \infty} f(x_n) \in \mathbb{R} \quad \forall f \in X^*.$$

2. $\{x_n\}$ **converges weakly** to $x \in X$ if

$$\lim_{n \rightarrow \infty} f(x_n) = f(x) \quad f \in X^*.$$

This is denoted by $x_n \rightharpoonup x$.

Example 9.2. Since $L^q = (L^p)^*$, we have the following concrete example. For $p \in (1, \infty)$, then $f_n \rightharpoonup f$ in L^p . Then for all $g \in L^q$, $\int f_n g \, d\mu \rightarrow \int f g \, d\mu$.

Theorem 9.3. Let X be a Banach space.

- If $x_n \rightharpoonup x$ in X , then $x_n \rightarrow x$ in X .
- If $x_n \rightharpoonup x$ in X then $\sup_n \|x_n\| < \infty$ and $\|x\| \leq \liminf_n \|x_n\|$.

Example 9.4 (Riemann-Lebesgue Lemma). Take $g \in C([0, 2\pi])$ and $f_n \in L^2([0, 2\pi])$, $f_n = \cos nx$. Then

$$\int_0^{2\pi} g f_n(x) \, dx = \int_0^{2\pi} g \cos nx \, dx \rightarrow 0,$$

which says $f_n \rightharpoonup 0$. Also $\int_0^{2\pi} (\cos nx)^2 \, dx = \pi$.

Proof of theorem. 1. For $f \in X^*$,

$$|f(x_n) - f(x)| = |f(x_n - x)| \leq \|f\| \|x_n - x\| \rightarrow 0$$

2. For $f \in X^*$, $f(x_n) \rightarrow f(x)$. Then $\{f(x_n)\}$ is bounded in \mathbb{R} . Now $\sup_n \|f(x_n)\| < \infty$. Recall that we can embed $x_n = \hat{x}_n$ in X^{**} . Then $\|f(x_n)\| = \|\hat{x}_n(f)\|$ and $\|x_n\| = \|\hat{x}_n\|$. By the uniform boundedness principle, it follows from $\sup_n \|\hat{x}_n(f)\| < \infty$ that $\sup_n \|x_n\| = \sup_n \|\hat{x}_n\| < \infty$.

Now

$$\begin{aligned} |\hat{x}(f)| &= |f(x)| \\ &= \lim_n |f(x_n)| \\ &\leq \|f\| \liminf_n \|x_n\|. \end{aligned}$$

Hence $\|x\| = \|\hat{x}\| \leq \liminf_n \|x_n\|$.

□

Definition 9.5. X is Banach. A sequence $\{f_n\} \subset X^*$ is called **weak* convergent** if

$$\lim_n f_n(x) \in \mathbb{R}, \quad \forall x \in X.$$

We say $f_n \rightarrow_{w^*} f$ if $\lim_n f(x) = f(x)$.

For reflexive spaces, this is the same as weak convergence.

Theorem 9.6. X is Banach and $f_n, f \in X^*$.

1. $f_n \rightarrow f$ implies $f_n \rightarrow_{w^*} f$
2. $f_n \rightarrow_{w^*} f$ implies $\sup \|f_n\| < \infty$ and $\|f\| \leq \liminf \|f_n\|$.

If X is reflexive ($X^{**} = X$). Then w^* convergence is the same thing as w convergence in X^* .

Theorem 9.7. X is Banach, $\{f_n\}, f \in X^*$. Then $f_n \rightarrow_{w^*} f$ if and only if $\sup \|f_n\| < \infty$. Also $\lim f_n(x) = f(x)$ if it holds for all $x \in D$, D dense in X .

Proof. \Rightarrow This is easy.

\Leftarrow Fix $x \in X$ and $\epsilon > 0$, and let $C = \sup \|f_n\| + \|f\|$. Take $x' \in D$ such that $\|x - x'\| < \frac{\epsilon}{3C}$.

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(x')| + |f_n(x') - f(x')| + |f(x') - f(x)| \\ &\leq \|f_n\| \|x - x'\| + |f_n(x') - f(x')| + \|f\| \|x - x'\|. \end{aligned}$$

Since $f_n(x') \rightarrow f(x')$, there is an $N \in \mathbb{N}$ such that for $n > N$, $|f_n(x') - f(x')| < \frac{\epsilon}{3}$. Thus the result follows. □

Recall Problem 2 from the 525 final: Given bounded functions in L^2 , there is a subsequence that converges weakly in L^2 .

Corollary 9.8. For $p \in (1, \infty)$, let $\{f_n\} \subset L^p(\mathbb{R}^d)$ such that $\sup \|f_n\|_p < \infty$. Then there is a subsequence $\{f_{n_k}\} \subset \{f_n\}$ which converge weakly in L^p .

Proof. **Step 1:** $L^q(\mathbb{R}^d) = (L^p(\mathbb{R}^d))^*$ is separable (take cutoffs of polynomials on balls). Take D to be the constructed countable dense subset.

Step 2: By diagonalization, construct $\{f_{n_k}\} \subset \{f_n\}$ such that

$$\int f_{n_k} g \rightarrow Tg \quad \forall g$$

Step 3: For $g \in L^q$, show that Tg is linear and bounded. Thus by Riesz Representation, we obtain an $f \in L^p$ such that $Tg = \int fg \, dx$. □

The most general result in this direction is Alaoglu's Theorem, which we won't prove since it is abstract and uses Tychonoff's Theorem. In addition, it requires defining the topology in terms of nets (we have only talked about sequences).

Theorem 9.9. If X is Banach, then the unit ball in X^* is compact in the weak* topology.

Lecture 10: April 21

Inequalities

Lemma 10.1 (Chebyshev's Inequality). *If $f \in L^p$, $p \in (0, \infty)$, and $\alpha > 0$ then*

$$\mu\{x: |f(x)| \geq \alpha\} \leq \left(\frac{\|f\|_p}{\alpha}\right)^p$$

Proof.

$$\begin{aligned} \|f\|_p^p &= \int |f|^p \, d\mu \\ &\geq \int_{|f(x)| \geq \alpha} \alpha^p \\ &= \alpha^p \mu\{x: |f(x)| \geq \alpha\} \end{aligned}$$

□

Let (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) be complete σ -finite spaces, and fix $p \in [1, \infty]$. Let $K: X \times Y \rightarrow \mathbb{R}$ be $\mathcal{M} \times \mathcal{N}$ -measurable. Suppose there is $C > 0$ such that

$$\begin{aligned} \int |K(x, y)| \, d\nu(y) &\leq C && \mu\text{-a.e. } x \\ \int |K(x, y)| \, d\mu(x) &\leq C && \nu\text{-a.e. } y \end{aligned}$$

Theorem 10.2 (Folland, 6.18). *If $f \in L^p(\nu)$, then*

1. *The integral $Tf(x) = \int K(x, y)f(y) \, d\nu(y)$ converges absolutely for μ -a.e. $x \in X$*
2. $Tf \in L^p(\mu)$
3. $\|Tf\|_{L^p(\mu)} \leq C\|f\|_{L^p(\nu)}$

Proof. Suppose $p \in (1, \infty)$. Write $|K(x, y)||f(y)| = |K(x, y)|^{1/q}|K(x, y)|^{1/p}|f(y)|$. Then by Hölder,

$$\int |K(x, y)||f(y)| \, d\nu(y) \leq C^{1/q} \left(\int |K(x, y)||f(y)|^p \, d\nu(y) \right)^{1/p}.$$

To show absolute convergence a.e., it suffices to show it belongs to $L^p(\mu)$. Taking p -norms,

$$\begin{aligned} \int \left[\int |K(x, y)| |f(y)| \, d\nu(y) \right]^p &\leq \int C^{p/q} \left(\int |K(x, y)| |f(y)|^p \, d\nu(y) \right) \\ &\leq C^{\frac{p}{q}+1} \|f\|_{L^p(\nu)}^p \end{aligned}$$

This required Tonelli, after which Fubini applies to yield $K(x, \cdot)f \in L^1(\nu)$ for a.e. x . Thus Tf is defined a.e., and $\|Tf\|_p \leq C\|f\|_p$ as desired.

The $p = 1$ doesn't even need Hölder; and is easier to prove. The $p = \infty$ case is trivial. \square

Minkowski's Inequality for Integrals

(X, \mathcal{M}, ν) and (Y, \mathcal{N}, μ) complete σ -finite, $p \in [1, \infty)$. $f: X \times Y \rightarrow \mathbb{R}$, $\mathcal{M} \times \mathcal{N}$ -measurable.

Theorem 10.3. *a. If $f \geq 0$ a.e. then*

$$\left[\int \left(\int f(x, y) \, d\nu(y) \right)^p \, d\mu(x) \right]^{1/p} \leq \int \left(\int f(x, y)^p \, d\mu(x) \right)^{1/p} \, d\nu(y)$$

b. If $p \in [1, \infty)$, $f(\cdot, y) \in L^p(\mu)$ for ν a.e. y , and $y \mapsto \|f(\cdot, y)\|_{L^p(\mu)}$ is ν measurable and in $L^1(\nu)$, then for μ a.e. x , $f(x, \cdot) \in L^1(\nu)$, and $x \mapsto \int f(x, y) \, d\nu(y)$ is in $L^p(\mu)$. Moreover,

$$\left\| \int f(\cdot, y) \, d\nu(y) \right\|_{L^p(\mu)} \leq \int \|f(\cdot, y)\|_{L^p(\mu)} \, d\nu(y).$$

Proof. For $p = 1$ we use Tonelli, and we compute with duality. Now $p \in (1, \infty)$. Take $g \in L^q(\mu)$. Then we compute by duality the p -norm as follows:

$$\begin{aligned} \int \left(\int f(x, y) \, d\nu(y) \right) |g(x)| \, d\mu(x) &= \int \left[\int f(x, y) |g(x)| \, d\mu(x) \right] \, d\nu(y) \\ &\leq \int \left[\left(\int |f(x, y)|^p \, d\mu(x) \right)^{1/p} \left(\int |g(x)|^q \, d\mu(x) \right)^{1/q} \right] \, d\nu(y) \\ &= \|g\|_{L^q(\mu)} \int \left[\left(\int |f(x, y)|^p \, d\mu(x) \right)^{1/p} \right] \, d\nu(y). \end{aligned}$$

Then (a) follows from [Folland, 6.14] and (b) follows from (a) and Fubini (except when $p = \infty$; then it is monotonicity of the integral). \square

The following inequality is useful on prelims, but be sure to state it correctly!

Theorem 10.4 (Folland, 6.20). *Let K be a Lebesgue measurable function on $(0, \infty) \times (0, \infty)$ such that $K(\lambda x, \lambda y) = \lambda^{-1} K(x, y)$ for all $\lambda > 0$ and*

$$\int_0^\infty |K(x, 1)| x^{-1/p} \, dx = C < \infty$$

for some $p \in [1, \infty]$. For $f \in L^p$ and $g \in L^q$, let

$$Tf(y) = \int_0^\infty K(x, y)f(x) dx, \quad Sg(x) = \int_0^\infty K(x, y)g(y) dy.$$

Then Tf and Sg are defined a.e., and $\|T\|, \|S\| \leq C$.

Corollary 10.5 (Folland, 6.21). Let

$$Tf(y) = y^{-1} \int_0^y f(x) dx, \quad Sg(x) = \int_x^\infty \frac{g(y)}{y} dy.$$

For $p \in (1, \infty]$ (so $q \in [1, \infty)$) we have

$$\|Tf\|_p \leq \frac{p}{p-1} \|f\|_p, \quad \|Sg\|_q \leq q \|g\|_q.$$

Proof. Just take

$$K(x, y) = \frac{\chi_{x < y}(x, y)}{y}$$

in the previous theorem. □

Lecture 11: April 23

Finishing the proof from last time

Theorem 11.1. $K: (0, \infty)^2 \rightarrow \mathbb{R}$ Lebesgue measurable, $p \in [1, \infty]$.

1. $K(\lambda x, \lambda y) = \lambda^{-1}K(x, y)$, $\lambda > 0$.

2. $\int_0^\infty |K(x, 1)|x^{-1/p} dx = C$

For $f \in L^p$, $g \in L^q$, define

$$Tf(y) = \int_0^\infty K(x, y)f(x) dx, \quad Sg(y) = \int_0^\infty K(x, y)g(y) dy.$$

Then $\|Tf\|_p \leq C\|f\|_p$ and $\|Sg\|_q \leq C\|g\|_q$.

Proof. Taking $x = zy$ and $f_z(y) = f(zy)$,

$$\begin{aligned} \int_0^\infty |K(x, y)||f(x)| dx &= \int_0^\infty |K(zy, y)||f(zy)|y dz \\ &= \int_0^\infty y^{-1}|K(z, 1)||f_z(y)|y dz \\ &= \int_0^\infty |K(z, 1)||f_z(y)| dz. \end{aligned}$$

Thus changing $x = zy$ yields

$$\begin{aligned} \|f_z\|_p &= \left(\int_0^\infty |f_z(y)|^p dy \right)^{1/p} \\ &= \left(\int_0^\infty |f(zy)|^p dy \right)^{1/p} \\ &= z^{-1/p} \left(\int_0^\infty |f|^p dx \right)^{1/p} \\ &= z^{-1/p} \|f\|_p. \end{aligned}$$

Then we use Minkowski to yield

$$\begin{aligned}
 \|Tf\|_p &= \left(\int_0^\infty \left(\int_0^\infty |K(z,1)| |f_z(y)| dz \right)^p dy \right)^{1/p} \\
 &\leq \int_0^\infty \left(\int_0^\infty |K(z,1)|^p |f_z(y)|^p dy \right)^{1/p} dz \\
 &= \int_0^\infty \left(|K(z,1)|^p \int_0^\infty |f_z(y)|^p dy \right)^{1/p} dz \\
 &= \int_0^\infty \left(|K(z,1)|^p \int_0^\infty |f_z(y)|^p dy \right)^{1/p} dz \\
 &= \int_0^\infty |K(z,1)| \|f_z\|_p dz \\
 &= \int_0^\infty |K(z,1)| z^{-1/p} \|f\|_p dz \\
 &= C \|f\|_p.
 \end{aligned}$$

Now we switch over to handle the Sg case. Switch via $x = y^{-1}$ and flipping bounds yields

$$\begin{aligned}
 \int_0^\infty |K(1,y)| y^{-1/q} dy &= \int_0^\infty y^{-1} |K(y^{-1},1)| y^{-1/q} dy \\
 &= \int_0^\infty x^{1+1/q} x^{-2} |K(x,1)| dx \\
 &= \int_0^\infty x^{-1/p} |K(x,1)| dx = C.
 \end{aligned}$$

□

Dual of L^∞

Consider the compact metric space (X, d) . Set $C(X) = \{f: X \rightarrow \mathbb{R} \text{ continuous}\}$. Let

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Remark. 1. $C(X)$ is a Banach space.

2. $\text{supp } f = \{x \in X: f(x) \neq 0\}$

3. $A, B \subset X$ closed and disjoint, then we have the continuous function

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \quad d(x, A) = \inf_{a \in A} d(x, a).$$

where $f = 0$ on A and $f = 1$ on B .

4. Partitions of unity: K compact, and $\{O_n\}$ an open cover. There are $\eta_k: X \rightarrow [0, 1]$ such that $\sum_{n=1}^m \eta_k = 1$ in K , where $\text{supp } \eta_k \subset O_k$.
5. μ is Borel regular on (X, d) if for all $E \in \mathcal{B}_X$ and for all $\epsilon > 0$, there is an open $U_\epsilon \supset E$ such that $\mu(U_\epsilon \setminus E) < \epsilon$. In addition, there is a compact $F_\epsilon \subset E$ such that $\mu(E \setminus F_\epsilon) < \epsilon$.

Clarification of Borel measures. If μ is a Borel measure on X , it is regular if for each ball B , $\mu(B) < \infty$. If μ is any measure on X , then μ is a metric measure if whenever $d(A, B) > 0$, we have $\mu(A \cup B) = \mu(A) + \mu(B)$ (the **Caratheodory condition**). We proved a while ago that if this condition holds, then μ is a Borel regular measure.

Candidate for the dual

Define

$$M(X) = \{\text{set of finite signed Borel regular measures on } X\}.$$

Given $\mu \in M(X)$, define $\|\mu\| = |\mu|(X)$. Then $(M(X), \|\cdot\|)$ is Banach. Indeed, it is linear and we have a norm. If $\|\mu\| = 0$, then $\mu^+(X) + \mu^-(X) = 0$ so each is zero. Triangle inequality follows by checking it for μ^+ and μ^- separately.

Suppose $\{\mu_n\} \subset M(X)$ converge absolutely, i.e. $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$. Then $\sum_{n=1}^{\infty} \mu_n^+(X) < \infty$ and same for $-$. Then $\sum_{n \geq 1} \mu_n^\pm$ are both Borel regular finite measures. To see this, we need to take intersections of open sets covering and unions of closed sets inside. Now take $\mu = \mu^+ - \mu^-$, and it remains to show $\|\mu - \sum_{n=1}^l \mu_n\| \rightarrow 0$ as $l \rightarrow \infty$.

Definition 11.2. $\ell: C(X) \rightarrow \mathbb{R}$ is called a **positive** linear functional if $\ell(f) \geq 0$ for all $f \geq 0$.

Remark. If ℓ is a positive linear functional, then $\|\ell\|_\infty < \infty$ and $\|\ell\| = \ell(1)$.

Proof. $|f(x)| \leq \|f\|$; thus $g = \|f\| - f \geq 0$, and $h = \|f\| + f \geq 0$. Thus $|\ell(f)| \leq \|f\|\ell(1)$; equality when $f = 1$. \square

Lecture 12: April 25

Theorem 12.1. Let X be a compact metric space, and ℓ a positive linear functional. There is a unique (positive) finite Borel measure μ such that

$$\ell(f) = \int f \, d\mu \quad \forall f \in C(X).$$

We start on the proof of existence.

For $V \subset X$ open, set

$$\rho(V) = \sup_{f \in C(X)} \ell(f), \quad 0 \leq f \leq 1 \text{ and } \text{supp } f \subset V.$$

Now for all $E \subset X$, set

$$\mu_* E = \int_{V \supset E} \rho(V), \quad V \text{ open.}$$

Claim 12.2. μ_* is a finite metric outer measure

This comprises the following:

(a) μ_* is an outer measure: $\mu_* \emptyset = 0$ and countable subadditivity

(b) μ_* is a metric measure:

$$\mu_*(E_1 \cup E_2) = \mu_* E_1 + \mu_* E_2, \quad d(E_1, E_2) > 0$$

Proof. Note that $\rho(X) = \ell(1)$ because for all $f \in C(X)$, $0 \leq f \leq 1$ we have

$$0 \leq \ell(f) \leq \|\ell\| \|f\| \leq \ell(1) < \infty.$$

Thus for $E \subset X$, we have $\mu_* E \leq \rho(X) < \infty$.

To prove (a), note that if V is open then $\mu_*(V) = \rho(V)$, and $\mu_*(\emptyset) = \rho(\emptyset) \leq \ell(0) = 0$. Now suppose that $E_1 \subset E_2$. Then for any open $O \supset E_2$, we have $O \supset E_1$. Thus from the infima definition, $\mu_* E_1 \leq \mu_* E_2$.

For countable subadditivity, take $\{V_j\}_{j=1}^{\infty}$ to be an open partition of V . Take $g \in C(X)$, with $0 \leq g \leq 1$ and $\text{supp } g \subset V$. Since X is compact, $\text{supp } g$ is compact. Take a partition of unity $\{\eta_i\}$ subordinate to $\{V_j\}$, and write $g = \sum_{i=1}^m \eta_i g$. Since $0 \leq \eta_i g \leq 1$ and $\text{supp } \eta_i g \subset V_i$,

$$\begin{aligned} \ell(g) &= \ell\left(\sum_{i=1}^m \eta_i g\right) \\ &= \sum_{i=1}^m \ell(\eta_i g) \\ &\leq \sum_{i=1}^m \rho(V_i) = \sum_{i=1}^m \mu_*(V_i) \\ &\leq \sum_{i=1}^{\infty} \mu_*(V_i). \end{aligned}$$

Hence $\mu_* V \leq \sum_{i=1}^{\infty} \mu_*(V_i)$.

Now we extend to countable subadditivity on all partitions; take $E = \bigcup_{k=1}^{\infty} E_k$. Then choose $U_k \supset E_k$ such that $\mu_* U_k \leq \mu_* E_k + \epsilon 2^{-k}$. Then $E \subset \bigcup_{k=1}^{\infty} U_k$, and

$$\mu_* E \leq \sum_{i=1}^{\infty} \mu_* E_i + \epsilon, \quad \forall \epsilon > 0.$$

Thus general countable subadditivity follows, completing the proof of (a).

For (b), start with E_1, E_2 such that $d(E_1, E_2) > 0$. Then there are open $O_i \supset E_i$ such that $d(O_1, O_2) > 0$ (take unions of open balls to get this). Take $g_i \in C(X)$ such that $0 \leq g_i \leq 1$ and $\text{supp } g_i \subset O_i$. Since $d(E_1, E_2) > 0$, it follows that $\text{supp } g_1 \cap \text{supp } g_2 = \emptyset$. Thus $0 \leq g_1 + g_2 \leq 1$ and consequently

$$\begin{aligned} \ell(g_1) + \ell(g_2) &= \ell(g_1 + g_2) \\ &\leq \mu_*(O_1 \cup O_2). \end{aligned}$$

Taking suprema over g_1, g_2 , it follows that $\mu_* O_1 + \mu_* O_2 \leq \mu_*(O_1 \cup O_2)$.

Choose an open $O \supset E_1 \cup E_2$. Then $E_i \subset O \cap O_i$, so

$$\begin{aligned} \mu_* E_1 + \mu_* E_2 &\leq \mu_*(O \cap O_1) + \mu_*(O \cap O_2) \\ &\leq \mu_* \{(O \cap O_1) \cup (O \cap O_2)\} \\ &\leq \mu_* O. \end{aligned}$$

□

Consequently, $\mu_*|_{\mathcal{B}_X} = \mu$ is a finite Borel regular measure such that $\mu(X) = \ell(1)$. It remains to show that for all $f \in C(X)$,

$$\int f \, d\mu = \ell(f).$$

Claim 12.3. For all $\epsilon > 0$ and open U , there is an $h \in C(X)$ such that

$$\ell(h) \leq \mu U \leq \ell(h) + \epsilon, \quad 0 \leq h \leq 1 \text{ and } \text{supp } h \subset U.$$

Proof. For every $\epsilon > 0$ and for all open U , there is a compact $K \subset U$ such that $\mu(U \setminus K) < \epsilon/2$. Then by Urysohn, there is an $h \in C(X)$ such that $0 \leq h \leq 1$, $h = 1$ on K , and $\text{supp } h \subset U$. By definition of μ , there is a $g \in C(X)$ such that $0 \leq g \leq 1$ and $\mu(U) \leq \ell(g) + \epsilon/2$. Note that $1 \geq g \geq h \geq 0$. Writing $g = h + (g - h)$, observe that $0 \leq g - h \leq 1$ and $\text{supp}(g - h) \subset U \setminus K$ (which is open). Then

$$\begin{aligned} \mu(U) &\leq \ell(g) + \epsilon/2 \\ &= \ell(h) + \epsilon/2 + \ell(g - h) \\ &\leq \ell(h) + \epsilon/2 + \mu(U \setminus K). \end{aligned}$$

Thus $\ell(h) \leq \mu(U) \leq \ell(h) + \epsilon$, as desired.

□

Lecture 13: April 28

Definition 13.1. A linear functional ℓ on $C(X)$ is called **positive** if

$$\ell(f) \geq 0 \quad \text{whenever } f \geq 0.$$

Let X be a compact metric space, and ℓ a positive linear functional on $C(X)$.

Theorem 13.2. *There is a unique Borel regular finite measure μ such that*

$$\ell(f) = \int f \, d\mu, \quad \forall f \in C(X).$$

Proof. Continued from last time. For open V , we set

$$\rho(V) = \sup\{\ell(f)\}, \quad \text{supremum taken over } f \in C(X), 0 \leq f \leq 1, \text{supp } f \subset V$$

Then we set $\mu_* E = \int \rho(V)$ over open $V \supset E$. Next we define $\mu = \mu_*|_{\mathcal{B}_X}$ (the Borel σ -algebra). We showed that μ is a finite regular Borel measure, and $\mu(X) = \ell(1)$. In addition, if U is open, then for any $\epsilon > 0$ there is a compact $K \subset U$ and $h \in C(X)$ such that $\mu(U \setminus K) < \epsilon$, $0 \leq h \leq 1$, $h = 1$ on K , $\text{supp } h \subset U$, and

$$\ell(h) \leq \mu(U) \leq \ell(h) + \epsilon.$$

Claim 13.3 (Existence).

$$\ell(f) = \int f \, d\mu, \quad \forall f \in C(X) \quad (*)$$

Note that both sides are linear; hence by the identity $f = f^+ - f^-$, it suffices to verify the result for $f \geq 0$. So assume without loss of generality that this is the case.

Fix an $\epsilon > 0$, and choose

$$0 = t_0 < t_1 < \dots < t_N, \quad t_{N-1} < \|f\|_\infty < t_N, \quad 0 < t_{i+1} - t_i < \epsilon, \quad \mu(f^{-1}(t_i)) = 0.$$

If $s \neq t$ then $f^{-1}(s) \cap f^{-1}(t) = \emptyset$. Hence there are only countable many s values such that $\mu(f^{-1}(s)) > 0$, since $\mu(X) < \infty$ (this is a general set theoretic principle; see [Folland, Chapter 0]).

Write $U_i = f^{-1}(t_{i-1}, t_i)$, which is open. Take compact $K_i \subset U_i$ such that $\mu(U_i \setminus K_i) < \frac{\epsilon}{2N}$ and $h_i \in C(X)$ such that $0 \leq h_i \leq 1$, $\text{supp } h_i \subset U_i$, $h_i = 1$ on K_i , and

$$\ell(h_i) \leq \mu(U_i) \leq \ell(h_i) + \frac{\epsilon}{2N}.$$

To prove (*), decompose

$$f = f \left(1 - \sum_{i=1}^N h_i \right) + \sum_{i=1}^N h_i f$$

and observe that

$$\begin{aligned} \ell \left[f \left(1 - \sum_{i=1}^N h_i \right) \right] &= \|f\|_\infty \ell \left[\frac{f}{\|f\|_\infty} \left(1 - \sum_{i=1}^N h_i \right) \right] \\ &\leq \|f\|_\infty \mu \left(\bigcup_{i=1}^N U_i \setminus K_i \right) \\ &= \|f\|_\infty \sum_{i=1}^N \mu(U_i \setminus K_i) \\ &< \frac{\epsilon}{2} \|f\|_\infty. \end{aligned}$$

In addition,

$$\ell \left[f \left(\sum_{i=1}^N h_i \right) \right] = \sum_{i=1}^N \ell(h_i f).$$

We sandwich the latter by

$$\begin{aligned} \sum_{i=1}^N \ell(t_{i-1} h_i) &\leq \sum_{i=1}^N \ell(h_i f) \leq \sum_{i=1}^N \ell(t_i h_i) \\ \sum_{i=1}^N t_{i-1} \left(\mu(U_i) - \frac{\epsilon}{2N} \right) &\leq \sum_{i=1}^N \ell(h_i f) \leq \sum_{i=1}^N t_i \mu(U_i) \\ \sum_{i=1}^N t_{i-1} \mu(U_i) - \|f\|_\infty \frac{\epsilon}{2} &\leq \sum_{i=1}^N \ell(h_i f) \leq \sum_{i=1}^N t_i \mu(U_i) \end{aligned}$$

Putting everything together, we obtain

$$\sum_{i=1}^N t_{i-1} \mu(U_i) - \|f\|_\infty \frac{\epsilon}{2} \leq \ell(f) \leq \sum_{i=1}^N t_i \mu(U_i) + \frac{\epsilon}{2} \|f\|_\infty.$$

But we also know that

$$\sum_{i=1}^N t_{i-1} \mu(U_i) \leq \int f \, d\mu = \sum_{i=1}^N \int_{U_i} f \, d\mu \leq \sum_{i=1}^N t_i \mu(U_i).$$

Therefore

$$\begin{aligned} \left| \ell(f) - \int f \, d\mu \right| &\leq \sum_{i=1}^N t_i \mu(U_i) - \sum_{i=1}^N t_{i-1} \mu(U_i) + \frac{\epsilon}{2} \|f\|_\infty \\ &\leq \sum_{i=1}^N (t_i - t_{i-1}) \mu(U_i) + \frac{\epsilon}{2} \|f\|_\infty \\ &\leq \epsilon \mu(X) + \frac{\epsilon}{2} \|f\|_\infty. \end{aligned}$$

Taking $\epsilon \rightarrow 0$, the result follows; we have computed $\ell(f)$.

Lastly, we show uniqueness of μ . So suppose that μ, ν are finite regular Borel measures such that $\int f \, d\mu = \int f \, d\nu$ for all $f \in C(X)$. It suffices to show that $\mu(O) = \nu(O)$ for all open O , by regularity.

Given $\epsilon > 0$, there is a compact $K \subset O$ such that $\mu(O \setminus K) < \epsilon$ and $\nu(O \setminus K) < \epsilon$. Take $h \in C(X)$ such that $\chi_K \leq h \leq \chi_O$. Then

$$\mu(K) \leq \int h \, d\mu \leq \mu(O) \implies \mu(O) - \epsilon \leq \int h \, d\mu \leq \mu(O),$$

and similarly for ν ; consequently $|\mu(O) - \nu(O)| \leq \epsilon$, and we are done. \square

Lecture 14: April 30

Theorem 14.1 (Riesz Representation Theorem). *Let X be a compact metric space, let $\ell: C(X) \rightarrow \mathbb{R}$ be a bounded linear operator. Then there is a finite signed Borel measure μ such that*

$$\ell(f) = \int f \, d\mu, \quad \forall f \in C(X).$$

Moreover, $\|\ell\| = \|\mu\| = \|\mu\|(X)$ and $(C(X))^* = M(X)$, the space of finite signed Borel measures.

Proposition 14.2. *Let X be a compact metric space, let ℓ be a bounded linear functional on $C(X)$. Then there are positive linear functionals ℓ^+, ℓ^- such that*

$$\ell = \ell^+ - \ell^-, \quad \|\ell\| = \|\ell^+\| + \|\ell^-\| = \ell^+(1) + \ell^-(1).$$

Proof of theorem using the proposition. If ℓ is a positive linear functional, then there is a positive finite measure μ such that

$$\int f \, d\mu = \ell(f) \quad 0 \leq f \leq 1, f \in C(X),$$

and $\|\ell\| = \ell(1) = \mu(X)$.

Then we get μ^\pm finite Borel measures with $\ell^\pm(f) = \int f \, d\mu^\pm$. Hence

$$\begin{aligned} \ell(f) &= \ell^+(f) - \ell^-(f) \\ &= \int f \, d\mu^+ - \int f \, d\mu^- \\ &= \int f \, d(\mu^+ - \mu^-). \end{aligned}$$

Set $\mu = \mu^+ - \mu^-$; then writing $|\mu| = \mu^+ + \mu^-$, we find

$$|\mu|(X) = \mu^+(X) + \mu^-(X) = \|\ell^+\| + \|\ell^-\| = \|\ell\|,$$

as desired.

For uniqueness, observe that if there are μ, ν satisfying the hypotheses, then

$$\begin{aligned} \int f \, d\mu^+ - \int f \, d\mu^- &= \int f \, d\nu^+ - \int f \, d\nu^- \\ \implies \int f \, d(\mu^+ + \nu^-) &= \int f \, d(\mu^- + \nu^+). \end{aligned}$$

Thus $d\mu^+ + d\nu^- = d\mu^- + d\nu^+$, so $\mu^+ - \mu^- = \nu^+ - \nu^-$ and we're done. so that □

Proof of the proposition. Consider $f \in C(X), f \geq 0$. Set

$$\ell^+(f) = \sup\{\ell(\phi) : \phi \in C(X), 0 \leq \phi \leq f\}.$$

Notice that $\ell^+(f) \geq 0$. Then taking $\phi = f$ yields $\ell(f) \leq \ell^+(f)$. Also $\ell^+(f) \leq \|\ell\| \|f\|$, since for any $\phi \leq f$ we have

$$|\ell(\phi)| \leq \|\ell\| \|\phi\| \leq \|\ell\| \|f\|.$$

Take $\alpha \in \mathbb{R}_{>0}$. It is clear that $\ell^+(\alpha f) = \alpha \ell^+(f)$.

Claim 14.3. For $f, g, \in C(X)$ with $f, g \geq 0$,

$$\ell^+(f + g) = \ell^+(f) + \ell^+(g).$$

Proof. For any $\epsilon > 0$, there are $\phi, \psi \in C(X)$ with $0 \leq \phi \leq f$ and $0 \leq \psi \leq g$ and

$$\ell(\phi) \leq \ell^+(f) \leq \ell(\phi) + \epsilon, \quad \ell(\psi) \leq \ell^+(g) \leq \ell(\psi) + \epsilon.$$

Thus

$$\begin{aligned} \ell^+(f) + \ell^+(g) &\leq \ell(\phi) + \ell(\psi) + 2\epsilon \\ &= \ell^+(f + g) + 2\epsilon. \end{aligned}$$

Hence $\ell^+(f) + \ell^+(g) \leq \ell^+(f + g)$.

For the other inequality, take $0 \leq \phi \leq f + g$. Set $\phi_1 = \min(\phi, f)$, which is continuous with $0 \leq \phi_1 \leq f$. Then choose $\phi_2 = \phi - \phi_1$, whereupon $0 \leq \phi_2 \leq g$ (just check both cases in the minimum). Then

$$\begin{aligned} \ell(\phi_1) + \ell(\phi_2) &= \ell(\phi) \\ &\leq \ell^+(f) + \ell^+(g). \end{aligned}$$

Taking the suprema over ϕ_1, ϕ_2 yields $\ell^+(f + g) \leq \ell^+(f) + \ell^+(g)$. \square

Now we extend ℓ^+ to be a linear functional on $C(X)$. For $f \in C(X)$, decompose $f = f^+ - f^-$ and set $\ell^+(f) = \ell^+(f^+) - \ell^+(f^-)$. We need to show that this is "well-defined". In other words,

Claim 14.4. If $f = f_1 - f_2 = h_1 - h_2$ for $f_i, h_i \geq 0 \in C(X)$, then $\ell^+(f_1) - \ell^+(f_2) = \ell^+(h_1) - \ell^+(h_2)$.

Proof. Since $f_1 + h_2 = f_2 + h_1 \geq 0$, $\ell^+(f_1) + \ell^+(h_2) = \ell^+(f_2) + \ell^+(h_1)$ and the result follows. \square

Claim 14.5. ℓ^+ is linear on $C(X)$.

Proof. Just expand; $\ell^+(f + g) = \ell^+((f + g)^+) - \ell^+((f + g)^-)$. Now write out both sides and you get two decompositions for $f + g$. \square

Lastly, we check that for $\alpha < 0$ we have $\ell^+(\alpha f) = -\alpha(\ell^+(f^-) - \ell^+(f^+)) = \alpha\ell^+(f)$ as required. Moreover,

$$\begin{aligned} \|\ell^+(f)\| &\leq \ell^+(f^+) + \ell^+(f^-) \\ &= \ell^+(\|f\|) \\ &\leq \|\ell^+\| \|f\| \\ &\leq \|\ell\| \|f\|, \end{aligned}$$

so it is a bounded linear operator (we don't need this yet).

Define $\ell^- = \ell^+ - \ell$. Taking $\phi \geq 0$, we get $\ell^-(\phi) = \ell^+(\phi) - \ell(\phi) \geq 0$. Then $\|\ell\| \leq \|\ell^+\| + \|\ell^-\|$ gives one direction.

For $0 \leq \phi \leq 1$, it follows that $|2\phi - 1| \leq 1$. Thus $\|\ell\| \geq \ell(2\phi - 1) = 2\ell(\phi) - \ell(1)$. Taking the supremum yields $\|\ell\| \geq 2\|\ell^+\| - \ell(1) = \ell^+(1)$. Hence $\|\ell\| \geq \|\ell^+\| + \|\ell^-\|$ as desired. \square

Lecture 15: May 2

Counterexample to homework problem: $f_n \in L^p(\mathbb{R})$ and $\sup \|f_n\|_p < \infty$, $\lim \int f_n = \lim \int f$, then $f_n \rightarrow f$ weakly in L^p .

It fails when $p = 1$; take $f_n = \chi_{[n, n+1]}$.

Radon

Definition 15.1. A Borel regular measure μ on \mathbb{R}^d is said to be **Radon** if for each compact $K \subset \mathbb{R}^d$, $\mu(K) < \infty$.

Where did we use compactness? Well, we needed to get the partition of unity to be finite.

Corollary 15.2 (Riesz Representation Theorem). *If ℓ is a positive linear functional on $C_c(\mathbb{R}^d)$, there exists a unique measure μ such that*

$$\ell(f) = \int f \, d\mu, \quad f \in C_c(\mathbb{R}^d).$$

Proof. Write $\mathbb{R}^d = \cup_{k=1}^{\infty} B(0, k)$. We need to do some smoothing to get it to work (because just using characteristic functions wouldn't be smooth enough).

Choose functions $\xi_k \in C_c(\mathbb{R}^d)$ with $\chi_{\overline{B_k}} \leq \xi_k \leq \chi_{B_{2k}}$, such that $\xi_k(x) = \xi_1\left(\frac{x}{2k}\right)$. Define $\ell_k: C(\overline{B_{2k}}) \rightarrow \mathbb{R}$ by $f \mapsto \ell_k(f) = \ell(f\xi_k)$. Then ℓ_k is positive, since if $f \geq 0$ then so is $f\xi_k$.

By Riesz Representation Theorem, there is a measure μ_k such that $\ell_k(f) = \int f \, d\mu_k$, for $f \in C(\overline{B_{2k}})$. Observe that $\mu_k = 0$ outside of $\overline{B_{2k}}$.

Now consider any $f \in C_c(\mathbb{R}^d)$. There is a j_0 such that $\text{supp } f \subset B(0, j)$, for $j \geq j_0$. Then $\ell(f) = \ell_{j_0}(f)$. On the other hand, it is $\ell_j(f)$. Hence by uniqueness in Riesz Representation, it follows $\mu = \mu_j$.

(Her proof: Assume they are not the same; then take a borel set on which one is larger than the other. Use Urysohn's Lemma to complete.)

Let $\mu = \mu_j$ on B_j . Then we take our set E , split it up, and add. □

Compact Operators

We want to study operators that preserve some nice features from \mathbb{R}^k , even in infinite dimensional spaces. All our previous definitions of "nice" operators included the identity map as an example. This is the first time that the identity isn't nice enough.

Definition 15.3. Let X, Y be Banach spaces, and $T: X \rightarrow Y$ a linear operator. T is **compact** when for every bounded sequence $\{u_k\} \subset X$, the set $\{Tu_k\} \subset Y$ is precompact.

Unrolling the definition, this means that there is a subsequence $\{u_{n_k}\}$ such that $\{Tu_{n_k}\}$ converges in Y .

Claim 15.4. T is continuous.

Proof. Let B be the unit ball. Then TB is precompact, so \overline{TB} is compact. Since we're in a metric space, \overline{TB} is therefore bounded. Hence TB is bounded as well, and consequently T is a bounded linear operator. \square

Let \mathcal{H} be an infinite dimensional Hilbert space. Then Id is not compact (since unit ball is not compact). But if $T: \mathcal{H} \rightarrow \mathbb{R}^d$ is any bounded linear map, then it is compact since the unit ball maps into a ball. (We could have worked with Banach instead of Hilbert, but we need Hilbert later on.)

Definition 15.5. When T is a linear map into a finite dimensional space, we say T is of **finite rank**.

Proposition 15.6. Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator, and \mathcal{H} is separable.

- 1) If $S: \mathcal{H} \rightarrow \mathcal{H}$ is compact, then ST and TS are compact.
- 2) If $\{T_n\}$ is a family of compact operators and $\|T_n - T\| \rightarrow 0$, then T is compact.
- 3) If T is compact, then there are a sequence of finite rank operators T_n such that $\|T_n - T\| \rightarrow 0$.
- 4) $T: \mathcal{H} \rightarrow \mathcal{H}$ is compact if and only if $T^*: \mathcal{H} \rightarrow \mathcal{H}$ is compact.

Lecture 16: May 5

Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator. Take \mathcal{H} to be a separable Hilbert space (we really need separability, it's not just ornamental).

Claim 16.1 (Property 1). *If $S: \mathcal{H} \rightarrow \mathcal{H}$ is compact, then so are ST and TS .*

For ST . Let $\{f_n\} \subset \mathcal{H}$ with $\sup_n \|f_n\| < \infty$. Since T is bounded, $\sup_n \|Tf_n\| < \infty$. Hence there are $\{f_{n_k}\} \subset \{f_n\}$ such that $\{STf_{n_k}\}$ converges. \square

For TS . As before consider $\{f_n\} \subset \mathcal{H}$. Since S is compact, there are $\{f_{n_j}\} \subset \{f_n\}$ such that $\{Sf_{n_j}\}$ converges. Since T is continuous, $\{TSf_{n_j}\}$ converges. \square

Claim 16.2 (Property 2). *If $T_n: \mathcal{H} \rightarrow \mathcal{H}$ is compact and $\|T_n - T\| \rightarrow 0$, then T is compact.*

Proof. Let $\{f_n\} \subset \mathcal{H}$ with $\sup_n \|f_n\| < \infty$. Then there is a subsequence $\{f_{n_k}\}$ such that $\{T_1 f_{n_k}\}$ converges. Now there is a subsequence $\{f_{n_k^2}\} \subset \{f_{n_k}\}$ such that $\{T_2 f_{n_k^2}\}$ converges. Continuing on, we obtain $\{f_{n_k^j}\} \subset \{f_{n_k^{j-1}}\}$ such that

$$\left\{ T_l f_{n_k^j} \right\} \text{ converges } \forall l \leq j.$$

Let $f_{n_k} = f_{n_k^k}$. We claim that $\{Tf_{n_k}\}$ converges.

Fix $\epsilon > 0$. There is an m_0 such that $\|T_m - T\| < \epsilon$ for $m \geq m_0$. Write $M = \sup \|f_n\| < \infty$. Fix m ; since $\{T_m f_{n_j}\}$ is Cauchy,

$$\begin{aligned} \|Tf_{n_k} - Tf_{n_j}\| &\leq \|Tf_{n_k} - T_m f_{n_j}\| + \|T_m f_{n_k} - T_m f_{n_j}\| + \|T_m f_{n_k} - Tf_{n_j}\| \\ &\leq \|T - T_m\| (\|f_{n_k}\| + \|f_{n_j}\|) + \|T_m f_{n_k} - T_m f_{n_j}\| \\ &< 2\epsilon M + \epsilon. \end{aligned}$$

\square

Claim 16.3 (Property 3). *If T is compact, then there are operators $\{T_n\}$ of finite rank such that $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Let $\{e_k\}$ be a basis for \mathcal{H} . Let P_n be projection onto $\langle e_k \rangle_{k \leq n}$, and Q_n the projection onto $\langle e_k \rangle_{k > n}$. Then $T_n = P_n T$ is of finite rank, and $I = P_n + Q_n$. We claim that $\|T_n - T\| \rightarrow 0$. Indeed, consider $0 \neq f \in \mathcal{H}$. Then

$$\begin{aligned} \|T_n f - Tf\|^2 &= \|Q_n Tf\|^2 \\ &= \sum_{k=n+1}^{\infty} \langle Tf, e_k \rangle^2 \\ &\leq \sum_{k=n}^{\infty} \langle Tf, e_k \rangle^2. \end{aligned}$$

Hence for all $f \in \mathcal{H}$,

$$\|Q_n T f\| \leq \|Q_{n-1} T f\| \leq \|Q_{n-1} T\| \|f\|,$$

whereupon $0 \leq \|Q_n T\| \leq \|Q_{n-1} T\|$. Hence we have a decreasing positive sequence, which must converge; let $c = \lim_n \|Q_n T\|$.

Suppose for contradiction that $c > 0$. Then there is a sequence $\{f_n\}$ with $\|f_n\| = 1$ such that $\|Q_n T f_n\| \geq c/2$. By compactness, there is a subsequence $\{T_{f_{n_k}}\}$ that converges to $g \in \mathcal{H}$. It follows that

$$\begin{aligned} \|Q_{n_k} T f_{n_k} - Q_{n_k} g\| &\leq \|Q_{n_k}\| \|T f_{n_k} - g\| \\ &\leq \|T f_{n_k} - g\|, \end{aligned}$$

which tends to 0. Hence there is a k_0 such that $\|Q_{n_k} g\| \geq c/4$, for all $k \geq k_0$. This is impossible, since

$$\|Q_{n_k} g\|^2 = \sum_{l \geq n_k+1}^{\infty} \langle g, e_l \rangle^2 \rightarrow 0.$$

□

Claim 16.4 (Property 4). $T: \mathcal{H} \rightarrow \mathcal{H}$ is compact precisely when $T^*: \mathcal{H} \rightarrow \mathcal{H}$ is compact.

Proof. Recall that $\langle T f, g \rangle = \langle f, T^* g \rangle$. Since T is compact, from the last claim $\|P_n T - T\| \rightarrow 0$. But

$$\begin{aligned} \|P_n T - T\| &= \|(P_n T)^* - T^*\| \\ &= \|T^* P_n^* - T^*\| \\ &= \|T^* P_n - T^*\|. \end{aligned}$$

However, $T^* P_n$ is compact. Thus T^* is a limit of compact operators, and is therefore compact. □

Examples from PDE

Studying compact operators is super important in PDE. Consider $f \in L^2(\mathbb{R}^d)$, and a measurable kernel $K(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$. Define the integral operator

$$Tf(x) = \int K(x, y) f(y) dy, \quad (*)$$

Consider $\Omega \subset \mathbb{C}^2$, and $\Omega \subset \mathbb{R}^n$. Consider the Laplacian $\nabla = \sum_i \frac{\partial^2}{\partial x_i^2}$. You want to solve the PDE $\nabla u = 0$, subject to either the Dirichlet boundary condition $u = f$ on $\partial\Omega$, or the Neumann boundary condition $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$.

It turns out that $\mu_f(x) = f(x) + Tf(x)$, in other words $I + T$.

To know if you can solve this function in L^2 , you need to prove that $Tf(x)$ is bounded in L^2 . You also care if it is invertible. Once it's compact, you have beautiful properties about the solution.

Definition 16.5. When $K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, the operator in (*) is a **Hilbert-Schmidt operator**

Lecture 17: May 7

Proposition 17.1. Let T be a Hilbert-Schmidt operator

$$Tf(x) = \int K(x, y)f(y) dy, \quad f \in L^2(\mathbb{R}^d), K \in L^2(\mathbb{R}^d \times \mathbb{R}^d).$$

1. If $f \in L^2(\mathbb{R}^d)$ for a.e. $x \in \mathbb{R}^d$, then

$$y \mapsto K(x, y)f(y)$$

is integrable.

2. The operator $T: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is bounded, with

$$\|T\| \leq \|K\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}.$$

Proof. By Fubini,

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K(x, y)^2 dy \right) dx < \infty.$$

Then for a.e. $x \in \mathbb{R}^d$,

$$\int K(x, y)^2 dy < \infty.$$

By Cauchy,

$$\int_{\mathbb{R}^d} |K(x, y)||f(y)| dy \leq \left(\int_{\mathbb{R}^d} |K(x, y)|^2 dy \right)^{1/2} \left(\int_{\mathbb{R}^d} |f(y)|^2 dy \right)^{1/2}.$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^d} |Tf(x)|^2 dx &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K(x, y)|^2 dy \int_{\mathbb{R}^d} |f(y)|^2 dy \right) dx \\ &= \|K\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2 \|f\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

To show that T is compact, we want to write it as a limit of finite rank operators. Thus we need to project it somehow onto a nice orthonormal basis for the product space $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. So start with $\{\phi_k\}$ an orthonormal basis on $L^2(\mathbb{R}^d)$. Take the functions $\{\phi_k(x)\phi_l(y)\}_{k,l}$; in the next homework, we'll prove that this is an orthonormal basis for $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

Thus we may write

$$K(x, y) = \sum_{k,l} a_{kl} \phi_k(x) \phi_l(y), \quad \sum_{k,l} |a_{kl}|^2 < \infty.$$

Define $K_n(x, y) = \sum_{k,l=1}^n a_{kl} \phi_k(x) \phi_l(y)$, and set

$$\begin{aligned} T_n f(x) &= \int K_n(x, y) f(y) dy \\ &= \sum_{k=1}^n \left[\int_{\mathbb{R}^d} \left(\sum_{l=1}^n a_{kl} \phi_l(y) f(y) \right) dy \right] \phi_k(x). \end{aligned}$$

Consequently $T_n f$ has finite rank.

Claim 17.2. $\|T_n - T\| \rightarrow 0$

Proof. Compute the tail

$$K - K_n = \sum_{\max(k,l) \geq n+1} a_{kl} \phi_k(x) \phi_l(y).$$

We can express its norm $\|K - K_n\|^2$ as a tail; indeed,

$$\begin{aligned} \sum_{k,l=1}^{\infty} |a_{kl}|^2 &= \sum_{k=1}^{\infty} \left(\sum_{l=1}^{\infty} |a_{kl}|^2 \right) < \infty, \\ \implies \sum_{k=n+1}^{\infty} \sum_{l=1}^{\infty} |a_{kl}|^2 &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Thus $\|K - K_n\|^2 \rightarrow 0$. Since $\|T - T_n\| \leq \|K - K_n\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$, the claim follows. \square

Now since limits of compact operators are compact, the result follows. \square

Theorem 17.3. Let $K: \mathcal{H} \rightarrow \mathcal{H}$ be a compact linear operator.

1. $\text{Ker}(I - K)$ is finite dimensional (and a priori closed)
2. $\text{Im}(I - K)$ is closed
3. $\text{Im}(I - K) = \text{Ker}(I - K^*)^\perp$
4. $\text{Ker}(I - K) = 0 \Leftrightarrow \text{Im}(I - K) = \mathcal{H}$
5. $\dim \text{Ker}(I - K) = \dim \text{Ker}(I - K^*)$

We have to make a leap of faith; lots of PDE operators can be written as $I - K$. So we will rewrite the theorem in different terms.

Fredholm Alternative

Your PDE question usually reduces to: given $f \in L^2(\mathbb{R}^d)$, can you find $u \in L^2(\mathbb{R}^d)$ such that $u - Ku = f$? The answer is yes if the kernel is 0.

Theorem 17.4. *Let K be compact on $\mathcal{H} = L^2(\mathbb{R}^d)$. Either:*

1. *For each $f \in \mathcal{H}$, there is a unique $u \in \mathcal{H}$ such that $u - Ku = f$*
2. *The space of homogeneous solutions to $u - Ku = 0$ is finite dimensional*

Let's assume we're in the case with non-trivial solutions; then the question ask, for which f can you solve the PDE? If f is in the image of the complement of the transpose, you can solve it by (3). So functional analysis informs PDE.

Proof. 1. Assume $\text{Ker}(I - K)$ is not finite. Since it is a Hilbert space, there is an orthonormal basis $\{u_k\}$. Then $\|u_k - u_j\|^2 = 2$, so it has no convergent subsequence. Since K is compact, there must be a convergent subsequence $Ku_{n_k} = u_{n_k}$ converges.

□

Lecture 18: May 9

Theorem 18.1. Let $K: \mathcal{H} \rightarrow \mathcal{H}$ be a compact linear operator.

1. $\text{Ker}(I - K)$ is finite dimensional (and a priori closed)
2. $\text{Im}(I - K)$ is closed
3. $\text{Im}(I - K) = \text{Ker}(I - K^*)^\perp$
4. $\text{Ker}(I - K) = 0 \Leftrightarrow \text{Im}(I - K) = \mathcal{H}$
5. $\dim \text{Ker}(I - K) = \dim \text{Ker}(I - K^*)$

Proof. 2. Take $v_k \in \text{Im}(I - K)$, with $v_k \rightarrow v$. Then there are w_k such that $w_k - Kw_k = v_k$. Since $\mathcal{H} = \text{Ker}(I - K) \oplus \text{Ker}(I - K)^\perp$, we can write $w_k = \tilde{w}_k + u_k$ for $\tilde{w}_k \in \text{Ker}(I - K)$ and $u_k \in \text{Ker}(I - K)^\perp$. Then $v_k = u_k - Ku_k$.

Claim 18.2.

$$\|u - Ku\| \geq \gamma \|u\|, \quad \forall u \in \text{Ker}(I - K)^\perp$$

for some $\gamma > 0$.

Proof of claim. Assume to the contrary that for each k , there is some $u_k \in \text{Ker}(I - K)^\perp$ such that

$$\frac{\|u_k\|}{k} > \|u_k - Ku_k\|.$$

Taking $v_k = \frac{u_k}{\|u_k\|}$, we find that $\|v_k - Kv_k\| < \frac{1}{k}$. Since $\{v_k\}$ is bounded, there is a convergent subsequence $\{v_{n_k}\} \rightarrow v$. By continuity of K ,

$$\begin{aligned} v - Kv &= \lim_k (v_{n_k} - Kv_{n_k}) \\ &= 0. \end{aligned}$$

Thus $v \in \text{Ker}(I - K)$. Since we started with $v_{n_k} \in \text{Ker}(I - K)^\perp$ and this set is closed, $v_{n_k} \in \text{Ker}(I - K)^\perp$ as well. Consequently $v = 0$, which contradicts continuity of $\|\cdot\|$. \square

Applying the claim, $\|v_k\| \geq \gamma \|u_k\|$. Since $\{v_k\}$ converges, it is bounded and hence $\{u_k\}$ is bounded. Since K is compact, Ku_{n_k} converges for some subsequence:

$$v_{n_k} + Ku_{n_k} = u_{n_k} \rightarrow u.$$

Then by continuity of K , $v_{n_k} \rightarrow u - Ku = v \in \text{Im}(I - K)$ and we're done.

3. If $T: \mathcal{H} \rightarrow \mathcal{H}$ is bounded and linear, then $\overline{\text{Im}(T)} = \text{Ker}(T^*)^\perp$. Indeed, suppose that $v \in \text{Ker}(T^*)$ and $u \in \text{Im}(T)$. Write $u = Tw$; then

$$\begin{aligned}\langle u, v \rangle &= \langle Tw, v \rangle \\ &= \langle w, T^*v \rangle \\ &= 0.\end{aligned}$$

Hence $\text{Im}(T) \subset \text{Ker}(T^*)^\perp$. But the latter set is closed, so $\overline{\text{Im}(T)} \subset \text{Ker}(T^*)^\perp$, as desired.

For the other direction, consider $v \in \overline{\text{Im}(T)}^\perp$. Consider any $u \in \mathcal{H}$. Then there are $v_n \in \text{Im}(T)^\perp$ such that $v_n \rightarrow v$. Consequently

$$\begin{aligned}0 &= \langle Tu, v_n \rangle \\ &\rightarrow \langle Tu, v \rangle.\end{aligned}$$

Thus $\langle u, T^*v \rangle = 0$ for every $v \in \overline{\text{Im}(T)}^\perp$. Hence $T^*v = 0$, whereupon $v \in \text{Ker } T^*$. Hence $\overline{\text{Im}(T)}^\perp \subset \text{Ker } T^*$, whereupon $\overline{\text{Im}(T)} \supset (\text{Ker } T^*)^\perp$.

4. \Rightarrow Assume that $\text{Ker}(I - K) = 0$, and that $\mathcal{H}_1 = \text{Im}(I - K) \neq \mathcal{H}$. Since \mathcal{H}_1 is closed (by 2), it's a sub-Hilbert space. Then there is some $u_0 \in \mathcal{H}$ with $\|u_0\| = 1$ and $u_0 \in \mathcal{H}_1^\perp$; in particular, $u_0 \notin \mathcal{H}_1$.

Consider the element $u_0 - Ku_0 \in \mathcal{H}_1$. We claim that $u_0 - Ku_0 \notin (I - K)\mathcal{H}_1 = \mathcal{H}_2$. Indeed, if $u_0 - Ku_0 = (I - K)h$ for some $h \in \mathcal{H}_1$, then $u_0 - h \in \text{Ker}(I - K)$. Hence $u_0 = h \in \mathcal{H}_1 \cap \mathcal{H}_1^\perp$, whereupon $u_0 = 0$ (contradiction).

It follows that $\mathcal{H}_2 \subsetneq \mathcal{H}_1 \subsetneq \mathcal{H} = \mathcal{H}_0$. Continuing inductively, we obtain $u_i \in \mathcal{H}_i$ such that $u_i \in \mathcal{H}_i^\perp$ and $\|u_i\| = 1$, where

$$\mathcal{H}_k = (I - K)^k \mathcal{H} \subsetneq \mathcal{H}_{k-1}.$$

If $k > l$, then $\mathcal{H}_k \subset \mathcal{H}_{l+1}$ so $\mathcal{H}_{l+1}^\perp \subset \mathcal{H}_k^\perp$. Since $u_l \in \mathcal{H}_l$, it follows that $u_l \in \mathcal{H}_k^\perp$. But then

$$\begin{aligned}K(u_k - u_l) &= (K - I)u_k + (I - K)u_l + u_k - u_l \\ &\in (\mathcal{H}_{k+1}) + (\mathcal{H}_{l+1}) + \mathcal{H}_{l+1} + \mathcal{H}_{l+1}^\perp.\end{aligned}$$

Therefore $\|K(u_k - u_l)\|^2 = \|\cdot\|^2 + \|u_l\|^2 \geq 1$. Hence we have constructed a sequence such that $\{Ku_k\}$ has no convergent subsequence, contradicting compactness of K .

- \Leftarrow Suppose $\text{Ker}(I - K^*) = 0$. Then since $(K^*)^* = K$ and K^* is compact, it follows from the previous half of 4 that $\text{Im}(I - K^*) = \mathcal{H}$. Hence $(\text{Ker}(I - (K^*)^*))^\perp = \mathcal{H}$, so that $\text{Ker}(I - K) = 0$.

□

Lecture 19: May 12

Recall part 5 from last time:

$$\dim \operatorname{Ker}(I - K) = \dim \operatorname{Ker}(I - K^*)$$

We only need to show one inequality, then the other follows by interchanging the roles of K, K^* . We also saw that $\operatorname{Ker}(I - K^*) = \operatorname{Im}(I - K)^\perp$.

Proof sketch. Assume to the contrary that $\dim \operatorname{Ker}(I - K) < \dim \operatorname{Im}(I - K)^\perp$. Then there is an operator $A: \operatorname{Ker}(I - K) \rightarrow \operatorname{Im}(I - K)^\perp$ that is injective; but by dimension, it can't be surjective. Then extend $A = 0$ on $\operatorname{Ker}(I - K)^\perp$. The contradiction arises since the operator $I - (K + A)$ (which is a sum of compact operators) behaves incorrectly. \square

Spectrum of a compact operator

Consider $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$.

Definition 19.1. 1. The **resolvent** of T is

$$\rho(T) = \{\eta \in \mathbb{R} \text{ (or } \mathbb{C}): T - \eta I \text{ is bijective}\}$$

2. The **spectrum** of T is

$$\sigma(T) = \rho(T)^c \quad (\text{either in } \mathbb{R} \text{ or } \mathbb{C})$$

3. $\eta \in \mathbb{C}$ is an **eigenvalue** for T if $\operatorname{Ker}(T - \eta I) \neq 0$. This means there is an eigenvector/eigenfunction $w \neq 0$ such that $Tw = \eta w$.

4. The **point spectrum** is

$$\sigma_p(T) = \{\eta \in \mathbb{C}: \eta \text{ is an eigenvalue}\}$$

Let \mathcal{H} be an infinite dimensional Hilbert space (this is not a linear algebra problem), and $K: \mathcal{H} \rightarrow \mathcal{H}$ is compact.

Theorem 19.2. 1. $0 \in \sigma(K)$

2. $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$

3. Either $\sigma(K) \setminus \{0\}$ is finite, or it converges to 0.

Remark. Consider K bounded, and $\{\lambda_n\} \subset \mathbb{C}$ such that there are $f_n \in \mathcal{H}$ such that $\|f_n\| = 1$ and $Kf_n = \lambda_n f_n$. Then λ_n is bounded.

Proof. 1. If $0 \notin \sigma(K)$ and K is bijective, then K^{-1} exists and is bounded (by open mapping). Hence we may write $I = K \circ K^{-1}$; the first is compact, and the second is bounded; thus I is compact, contradiction.

2. We show the contrapositive; consider $\eta \neq 0$ such that $\eta \notin \sigma_p(K)$. Then $\text{Ker}(K - \eta I) = 0$. Since $\eta \neq 0$, we can divide and apply the Fredholm Alternative to

$$\text{Ker}\left(I - \frac{1}{\eta}K\right) = 0.$$

Consequently $\text{Im}(K - \eta I) = \mathcal{H}$. It follows that $\eta \notin \sigma(K)$.

3. Assume that $\sigma(K)$ is not finite. Let $\{\eta_k\} \subset \sigma(K) \setminus \{0\}$ be distinct. Since $\{\eta_k\}$ is bounded, it converges to η . We claim that $\eta = 0$; then the result follows (by the infinite pigeonhole principle).

For each k , choose ω_k such that $\|\omega_k\| = 1$ and $K\omega_k = \eta_k\omega_k$. Set $H_k = \langle \omega_1, \dots, \omega_k \rangle$. We claim that $\{\omega_k\}$ is linearly independent.

Write $\omega_k = \sum_{j \leq k-1} \alpha_j \omega_j$ and proceed inductively. Then

$$\begin{aligned} K\omega_k &= \eta_k\omega_k \\ &= \sum_{j=1}^{k-1} \alpha_j \eta_j \omega_j \\ \implies \eta_k \sum_{j=1}^{k-1} \alpha_j \omega_j &= \sum_{j=1}^{k-1} \alpha_j \eta_j \omega_j. \end{aligned}$$

Hence by linear independence of $\{\omega_i\}_{i=1}^{k-1}$, it follows that each $\alpha_j = 0$ so they are linearly independent.

Thus $H_k \not\subset H_{k+1}$, and $(K - \eta_k I)H_k \subset H_{k-1} \subset H_k$. So for $k > l$, we get $H_{l-1} \subset H_l \subset H_{k-1} \subset H_k$. She wrote

$$H_k = H_{k-1} \oplus H_{k-1}^\perp$$

Let $u_k \in H_{k-1}^\perp \cap H_k$ with $\|u_k\| = 1$. Then

$$\begin{aligned} \left\| \frac{Ku_k}{\eta_k} - \frac{Ku_l}{\eta_l} \right\|^2 &= \left\| \frac{(K - \eta_k)u_k}{\eta_k} - \frac{(K - \eta_l)u_l}{\eta_l} + u_k - u_l \right\|^2 \\ &\geq 1, \end{aligned}$$

since $u_k \in H_{k-1}^\perp$ (and the rest is in H_{k-1}). Hence we've shown that $\{K(u_k)/\eta_k\}$ does not have a convergent subsequence. This contradicts compactness of K (since $\{u_k/\eta_k\}$ is bounded).

□

Definition 19.3. An operator T is **symmetric** if $T = T^*$.

Theorem 19.4 (Spectral Theorem). *If $T \neq 0$ is compact and symmetric on \mathcal{H} , then T can be diagonalized.*

Concretely, this means that there is a basis $\{\phi_k\}$ of eigenfunctions T such that

$$T\phi_k = \lambda_k\phi_k, \quad \lambda_k \in \mathbb{R}, \quad \lambda_k \rightarrow 0.$$

We make the following claims:

1. Every eigenvalue is real (only needs symmetric, not compact)
2. If ϕ_i, ϕ_j are eigenfunctions corresponding to different eigenvalues, then $\langle \phi_i, \phi_j \rangle = 0$.
3. Either $\|T\|$ or $-\|T\|$ is an eigenvalue for T .
4. $\mathcal{H} = \overline{\{\{\phi_k\}\}}$

Lecture 20: May 14

Theorem 20.1 (Spectral Theorem). *If $T: \mathcal{H} \rightarrow \mathcal{H}$ is compact and symmetric ($T = T^*$), then T can be diagonalized.*

Claim 20.2. *All eigenvalues of T are real.*

Proof. Assume $Tu = \lambda u$ with $\|u\| = 1$. Then

$$\lambda - \bar{\lambda} = (Tu, u) - (u, Tu) = (Tu, u) - (u, T^*u) = 0.$$

Hence $\lambda \in \mathbb{R}$. □

Claim 20.3. *If u_1, u_2 correspond to different eigenvalues λ_1, λ_2 then $(u_1, u_2) = 0$.*

Proof.

$$\begin{aligned} (\lambda_1 - \lambda_2)(u_1, u_2) &= (Tu_1, u_2) - (u_1, Tu_2) \\ &= (Tu_1, u_2) - (u_1, T^*u_2) \\ &= (Tu_1, u_2) - (Tu_1, u_2) = 0. \end{aligned}$$

Thus $(u_1, u_2) = 0$. □

Claim 20.4. *Either $\|T\|$ or $-\|T\|$ is an eigenvalue for T .*

The reason why we claim 3 is so that we know we have an eigenvalue in claim 4. Notice that we're working over the complex, but T is symmetric. Thus $(Tf, f) \in \mathbb{R}$ for all f .

Proof of claim. Write

$$\begin{aligned} \|T\| &= \sup\{|(Tf, f)|: \|f\| = 1\} \\ &= \begin{cases} \sup\{(Tf, f): \|f\| = 1\} \\ \sup\{-(Tf, f): \|f\| = 1\} \end{cases} \quad \text{or} \end{aligned}$$

Consider the first case (the second case goes similarly). Then there are f_n such that $\|f_n\| = 1$ such that $(Tf_n, f_n) \rightarrow \|T\|$. Hence there is a subsequence $Tf_{n_k} \rightarrow g \in \mathcal{H}$. We claim that g is an eigenfunction, and moreover $Tg = \lambda g$ where $\lambda = \|T\|$.

Indeed, observe that

$$\begin{aligned} \|Tf_n - \lambda f_n\|^2 &= \|Tf_n\|^2 - 2\lambda(Tf_n, f_n) + \lambda^2 \\ &\leq \|T\|^2\|f_n\|^2 - 2\lambda(Tf_n, f_n) + \lambda^2, \end{aligned}$$

which tends to $\lambda^2 - 2\lambda^2 + \lambda^2 = 0$ as $n \rightarrow \infty$. Thus

$$\lim_{n \rightarrow \infty} \|Tf_n - \lambda f_n\| = 0.$$

Now since $Tf_{n_k} \rightarrow g$ and $Tf_{n_k} - \lambda f_{n_k} \rightarrow 0$, it follows that $\lambda f_{n_k} \rightarrow g$. Taking norms, $\|g\| = \|T\|$. Then by continuity of T , we obtain $T(\lambda f_{n_k}) \rightarrow Tg$. But the former tends to λg , whereupon $\lambda g = Tg$. Hence g is an eigenfunction, with eigenvalue $\|T\|$. \square

Claim 20.5. Consider the linear subspace

$$S = \overline{\{\text{all eigenfunctions of } T\}} = \overline{\{\phi_k\}}.$$

Then $S = \mathcal{H}$.

Proof. Suppose that $S \subsetneq \mathcal{H}$. Then write $\mathcal{H} = S \oplus S^\perp$. If $f \in S$, then $Tf \in S$. Also if $g \in S^\perp$ and $f \in S$, then $(f, Tg) = (Tf, g) = 0$. Thus T restricts to the following operator:

$$T_1 = T|_{S^\perp}: S^\perp \rightarrow S^\perp.$$

Observe that the targets are both Hilbert spaces, and $T_1 \neq 0$ is compact and symmetric. Thus $\lambda_1 = \pm \|T_1\| \neq 0$ is an eigenvalue. Hence there is an eigenfunction $g \in S^\perp$ such that $\|g\| = 1$ and $T_1 g = \lambda_1 g$. But from the homework, the eigenvalues of T_1 must tend to 0; this is a contradiction. \square

Fourier Analysis

Notation

$E \subset \mathbb{R}^n$, $L^p(E) = L^p(E, L^n)$. If $U \subset \mathbb{R}^n$ is open, then $C^k(U) = \{f: U \rightarrow \mathbb{R}\}$ such that f and all partial derivatives up to order k are continuous. We set

$$C^\infty(U) = \bigcap_{k=1}^{\infty} C^k(U),$$

the smooth functions. Also $C_c^\infty(U) = C^\infty(U) \cap C_c$, and when $U = \mathbb{R}^n$ we drop the U dependence from our notation.

We set $\partial_j = \frac{\partial}{\partial x_j}$ for $j \leq n$.

Definition 20.6. A **multi-index** is an ordered n -tuple of non-negative integers. If $\alpha = (\alpha_1, \dots, \alpha_n)$, then its **length** is

$$|\alpha| = \sum_{i=1}^n \alpha_i,$$

and its factorial is

$$\alpha! = \prod_{j=1}^n \alpha_j!$$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set

$$\partial^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}, \quad x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}.$$

Assume that $f \in C^k$. Then Taylor's formula becomes

$$f(x) = \sum_{|\alpha| \leq k} \partial^\alpha f(x_0) \frac{(x - x_0)^\alpha}{\alpha!} + R_k(x), \quad \lim_{x \rightarrow x_0} \frac{|R_k(x)|}{|x - x_0|} = 0.$$

$f, g \in C^k$ $|\alpha| \leq k$ then

$$\partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} \partial^\beta f \partial^\gamma g$$

Lecture 21: May 16

Example 21.1.

For $t \in \mathbb{R}$, set

$$\eta(t) = e^{-1/t} \chi_{(0,\infty)} \in C^\infty.$$

Then for $x \in \mathbb{R}^d$, set

$$\Psi(x) = \eta(1 - |x|^2) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right), & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

Observe that $\Psi \in C^\infty(\mathbb{R}^d)$, and $\text{supp } \Psi = \overline{B_1}$.

Let \mathcal{S} be the Schwartz class (smooth functions such that all derivatives vanish at ∞ “faster than any polynomial”). For instance, e^{-x^2} .

For any $N \in \mathbb{N}$ and any multi-index α , we set

$$\|f\|_{N,\alpha} = \sup_{x \in \mathbb{R}^d} (1 + |x|)^N |\partial^\alpha f(x)|.$$

Formally, we define the Schwartz class as follows:

$$\mathcal{S} = \{f \in C^\infty : \|f\|_{N,\alpha} < \infty, \quad \forall N, \alpha\}.$$

Observe that $C_c^\infty(\mathbb{R}^d) \subsetneq \mathcal{S}$.

Remark. 1. If $f \in \mathcal{S}$, then $\partial^\alpha f \in L^p(\mathbb{R}^d)$ for all α . Thus $\|f\|_{N,\alpha} < C_{N,\alpha}$. Hence $|\delta^\alpha f(x)| \leq C_{N,\alpha} (1 + |x|)^{-N}$. Integrating,

$$\begin{aligned} \int_{\mathbb{R}^d} |\delta^\alpha f(x)|^p &\leq C_{N,\alpha}^p \int_{\mathbb{R}^d} (1 + |x|)^{-Np} dx \\ &\leq C_{N,\alpha} \left(\int_{B_1} (1 + |x|)^{-Np} dx + \int_{|x| \geq 1} (1 + |x|)^{-Np} dx \right) \\ &\leq C_{N,\alpha} |B_1| + (< \infty), \end{aligned}$$

for $N > \frac{d}{p}$.

Proposition 21.2. Let $\{f_n\} \subset \mathcal{S}$ such that

Lecture 22: May 19

Claim 22.1. $\partial^\alpha(x^\beta f)$ is bounded for all α, β implies that $x^\beta \partial^\alpha f$ is bounded for all α, β .

Proof. Expand as a finite linear combination. □

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ for $y \in \mathbb{R}^n$ set $\tau_y f(x) = f(x - y)$. Then $\|\tau_y f\|_p = \|f\|_p$ for $p < \infty$ and $\|\tau_y f\|_\infty = \|f\|_\infty$.

Definition 22.2. f is **uniformly continuous** whenever $\|\tau_y f - f\|_\infty \rightarrow 0$ as $|y| \rightarrow 0$.

Lemma 22.3. If $f \in C_c(\mathbb{R}^n)$ then f is uniformly continuous.

Proposition 22.4. If $1 \leq p < \infty$ then translation is continuous in L^p .

Proof. Apply with $\tau_z f$ in place of f ; then it suffices to show that $\|\tau_y f - f\|_p \rightarrow 0$. If $g \in C_c(\mathbb{R}^n)$, then $\|\tau_y g - g\|_p \rightarrow 0$ as $|y| \rightarrow 0$.

If $|y| < 1$, then $|\tau_y g - g|$ is supported in $\overline{\text{supp } g + B} = K$.

$$\int |\tau_y g - g|^p dx \leq \mu \leq \|\tau_y g - g\|_\infty^p m(K).$$

Now by uniform continuity, this tends to 0.

Consider any $f \in L^p$. Fix an $\epsilon > 0$, and choose $g \in C_c$ such that $\|f - g\|_p < \epsilon$. Choose δ such that for $|y| < \delta$, $\|\tau_y g - g\|_p < \epsilon$. Then $\|\tau_y f - f\|_p \leq 3\epsilon$, so we're done. □

Smoothing operators and convolution

Definition 22.5. If f, g are measurable on \mathbb{R}^n , then the **convolution** is

$$f * g(x) = \int f(x - y)g(y) dy.$$

when the integral exists.

Assuming all integrals that follow actually exist,

Proposition 22.6. 1. $f * g = g * f$

2. $(f * g) * h = f * (g * h)$

3. For $z \in \mathbb{R}^n$, $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$

4. $\text{supp}(f * g) \subset \overline{\text{supp } f + \text{supp } g}$

If $f \in L^1$ and $g \in L^\infty$, then $f * g$ is bounded. The idea is that even when you integrate something ugly, it becomes nice.

Proof. We need Fubini for (2). To prove (3),

$$\begin{aligned}\tau_z(f * g)(x) &= (f * g)(x - z) \\ &= \int f(x - z - y)g(y) \, dy \\ &= \int \tau_z f(x - y)g(y) \, dy \\ &= (\tau_z f) * g(x)\end{aligned}$$

(4) is clear, because the only way for $g(y)f(x - y) \neq 0$ is for $y \in \text{supp } g, x - y \in \text{supp } f$ whereupon $\text{supp}(f * g) \subset \text{supp } f + \text{supp } g$. \square

This is all fine and dandy, but so far we don't know if any of these integrals ever exist.

Theorem 22.7 (Young's Inequality). *If $f \in L^1$ and $g \in L^p$, then*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Choose $d\nu(y) = |f(y)| \, dy$; then by Minkowski,

Proof.

$$\begin{aligned}\int |f * g|^p \, dx &\leq \int \left(\int |f(y)||g(x - y)| \, dy \right)^p \, dx \\ &\leq \left(\int \left(\int |g(x - y)|^p \, dx \right)^{1/p} |f(y)| \, dy \right)^p \\ &\leq (\|g\|_p \|f\|_1)^p\end{aligned}$$

\square

Proposition 22.8. *If $f \in L^p$ and $g \in L^q$, then $f * g$ exists for every x ; also $f * g$ is bounded and uniformly continuous, with $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$. Lastly when $p \in (1, \infty)$, $f * g \in C_0(\mathbb{R}^n)$.*

In other words, convolution is a smoothing operator.

Proof. By Hölder, $|f * g|(x) \leq \|f\|_p \|g\|_q$. The uniform bound follows, as well as existence of $f * g$ for every x .

For uniform continuity,

$$\begin{aligned}\|\tau_y(f * g) - (f * g)\|_\infty &\leq \|(\tau_y f - f) * g\|_\infty \\ &\leq \|\tau_y f - f\|_p \|g\|_q.\end{aligned}$$

Hence for $1 \leq p < \infty$, we have the result. For $p = \infty$, interchange the roles of f, g .

Recall that $C_0(\mathbb{R}^n)$ is the closure of $C_c(\mathbb{R}^n)$ (in the uniform norm). Since $f \in L^p$ and $g \in L^q$, there are $f_k, g_k \in C_c$ such that $\|f_k - f\|_p \rightarrow 0$ and same for g . Then $\text{supp } f_k * g_k$ is compact. Lastly

$$\|f_k * g - f * g\|_u \leq \|f_k\|_p \|g_k - g\|_q + \|f_k - f\|_p \|g\|_q.$$

Since $\|f_k\|_p$ converges (and hence is bounded), the result follows. \square

Sneak peek of homework: Another version of Young's Inequality.

Proposition 22.9. *Consider $1 \leq p, q, r \leq \infty$ with $p^{-1} + q^{-1} = 1 + r^{-1}$. Then $\|f * g\|_r \leq \|f\|_p \|g\|_q$ whenever things are defined.*

Lecture 23: May 21

Convolution is a smoothing operator.

Proposition 23.1. *If $f \in L^1$, $g \in C^k$, and $\partial^\alpha g \in L^\infty$ for $|\alpha| \leq k$, then*

$$f * g \in C^k \text{ with } \partial^\alpha(f * g) = f * \partial^\alpha g.$$

Proof.

$$\begin{aligned} \frac{f * g(x + te_j) - f * g(x)}{t} &= \frac{1}{t} \int (f(y)g(x + te_j - y) - f(y)g(x - y)) \, dy \\ &\rightarrow \int f(y) \partial_j g(x - y) \, dy \\ &= f * \partial_j g(x), \end{aligned}$$

by dominated convergence. Indeed, by the mean value theorem

$$\frac{|g(x - y + te_j) - g(x - y)|}{|t|} \leq \|\partial_j g\|_\infty$$

so the integrand is bounded by $\|\partial_j g\|_\infty |f(y)|$ which is integrable. \square

Note: More generally, we may assume that $f \in L^p$ and $\partial^\alpha g \in L^q$ to obtain the same result via Hölder.

Proposition 23.2. *If $f, g \in \mathcal{S}$, then $f * g \in \mathcal{S}$.*

Proof. We show that $f * g \in C^\infty$. Note that $1 + |x| \leq 1 + |x - y| + |y| \leq (1 + |x - y|)(1 + |y|)$. Then

$$\begin{aligned} (1 + |x|)^N |\partial^\alpha(f * g)(x)| &\leq (1 + |x|)^N \int |\partial^\alpha f(x - y)| |g(y)| \, dy \\ &\leq \int (1 + |x - y|)^N |\partial^\alpha f(x - y)| (1 + |y|)^{N+n+1} (1 + |y|)^{-n-1} |g(y)| \, dy \\ &\leq \|f\|_{(N,\alpha)} \|g\|_{(N+n+1,0)} \int \frac{dy}{(1 + |y|)^{n+1}}. \end{aligned}$$

\square

For any $\Phi \in L^1$ with $\int \Phi \, dx = a$, set $\Phi_t(x) = t^{-n} \Phi(x/t)$, $t < 1$.

Theorem 23.3. *With Φ as above,*

1. *If $f \in L^p$, $p \in [1, \infty)$ then $f * \Phi_t \rightarrow af$ in L^p .*

2. If f is bounded and uniformly continuous, then $f * \Phi_t \rightarrow af$ uniformly.
3. If $f \in L^\infty$ and $f \in C(U)$, U open, then $f * \Phi_t \rightarrow af$ uniformly on compact sets.

Proof. In all cases,

$$\begin{aligned}
 f * \Phi_t(x) - af(x) &= \int (f(x-y) - f(x))\Phi_t(y) dy \\
 &= \int (f(x-y) - f(x))t^{-n}\Phi(y/t) dy \\
 (y = tz) \quad &= \int (f(x-yz) - f(x))\Phi(z) dz \\
 &= \int (\tau_{tz}f(x) - f(x))|\Phi(z)| dz.
 \end{aligned}$$

1. By the above,

$$\begin{aligned}
 \|f * \Phi_t - af\|_p &\leq \left(\int \left(\int |\tau_{tz}f(x) - f(x)| |\Phi(z)| dz \right)^p dx \right)^{1/p} \\
 \text{(Minkowski)} \quad &\leq \int \left(\int |\tau_{tz}f(x) - f(x)|^p dx \right)^{1/p} |\Phi(z)| dz \\
 &= \int \|\tau_{tz}f - f\|_p |\Phi(z)| dz.
 \end{aligned}$$

This tends to 0 by continuity of translation, $\|\tau_{tz}f\|_p = \|f\|_p$, and dominated convergence.

2. Observe that

$$\begin{aligned}
 |f * \Phi_t(x) - af(x)| &\leq \int |\tau_{tz}f(x) - f(x)| |\Phi(z)| dz \\
 &\leq \int \|\tau_{tz}f - f\|_u |\Phi(z)| dz \\
 \implies \|f * \Phi_t - af\|_u &\leq \int \|\tau_{tz}f - f\|_u |\Phi(z)| dz.
 \end{aligned}$$

Since f is uniformly continuous and $\Phi \in L^1$, the right side tends to 0 so by dominated convergence we're done.

3. We need to address that the integral involves things outside of U . We can cover most of U with a compact set, where the convergence is uniform. On the remaining portion, we can make Φ small.

Given $\epsilon > 0$, there is a compact $E \subset \mathbb{R}^n$ (it's a ball) such that

$$\int_{E^c} |\Phi| dx < \epsilon.$$

To prove uniform convergence, fix any compact $K \subset U$. Choose $\delta > 0$ such that for $|t| < \delta$, $K - tE \subset K' \subset U$ with K' compact. Then

$$\sup_{x \in K, z \in E} |f(x - yz) - f(x)| < \epsilon.$$

Hence

$$\begin{aligned} \sup_{x \in K} |f * \Phi_t(x) - af(x)| &\leq \sup_{x \in K} \int |f(x - tz) - f(x)| |\Phi(z)| dz \\ &= \sup_{x \in K} \left\{ \int_E |f(x - yz) - f(x)| |\Phi(z)| dz \right. \\ &\quad \left. + \int_{E^c} |f(x - yz) - f(x)| |\Phi(z)| dz \right\} \\ &\leq \int_E \sup_{x \in K, z \in E} |f(x - tz) - f(x)| |\Phi(z)| dz \\ &\quad + 2\|f\|_\infty \int_{E^c} |\Phi(z)| dz \\ &\leq \epsilon \int_E |\Phi(z)| dz + 2\|f\|_\infty \int_{E^c} |\Phi(z)| dz \end{aligned}$$

□

Lecture 24: May 23

Convolutions smooth things out.

Theorem 24.1. Suppose $|\phi(x)| \leq C(1+|x|)^{-n-\epsilon}$ for $\epsilon, C > 0$. Then $\phi \in L^1$ and assume that $\int \phi \, dx = \alpha$. If $f \in L^p$ for $p \in [1, \infty]$, $f * \phi_t(x) \rightarrow \alpha f$ for a.e. $x \in \mathbb{R}^n$.

Proof. The Lebesgue set of f consists of those points where

$$\int_{B(x,r)} |f(y) - f(x)| \, dy \rightarrow 0$$

as $r \rightarrow 0$. For all $\delta > 0$, there is a $\eta > 0$ such that for $0 < r < \eta$,

$$\int_{|y| < r} |f(x-y) - f(x)| \, dy \leq \delta m(B_r)$$

$$\begin{aligned} |f * \phi_t(x) - \alpha f(x)| &\leq \int_{|y| \leq \eta} |f(x-y) - f(x)| |\phi_t(y)| \, dy \\ &\quad + \int_{|y| > \eta} |f(x-y) - f(x)| |\phi_t(y)| \, dy \\ \int_{|y| > \eta} |f(x-y) - f(x)| |\phi_t(y)| \, dy &\leq \int_{|y| > \eta} (|f(x-y)| + |f(x)|) |\phi_t(y)| \, dy \\ &\leq |f(x)| \int_{|y| \geq \eta} |\phi_t(y)| \, dy + \|f\|_p \left(\int_{|y| \geq \eta} |\phi_t(y)|^q \, dy \right)^{1/q} \\ &= I + I_q. \end{aligned}$$

We estimate $\left(\int_{|y| \geq \eta} |\phi_t(y)|^q \right)^{1/q}$ in the case $q = \infty$.

$$\begin{aligned} |\phi_t(y)| &= t^{-n} |\phi(y/t)| \leq ct^{-n} \left(1 + \frac{|y|}{t} \right)^{-n-\epsilon} \\ &\leq ct^{-n} (1 + \eta/t)^{-n-\epsilon} = ct^\epsilon (t + \eta)^{-n-\epsilon} \\ &\leq c\eta^{-n-\epsilon} t^\epsilon \rightarrow 0 \quad (t \rightarrow 0). \end{aligned}$$

□

Thus we've shown $\|\phi_t(y)\chi_{|y| \geq \eta}\|_\infty \rightarrow 0$, so $I_\infty(t) \rightarrow 0$.

Now consider $q < \infty$. it follows that

$$\begin{aligned}
 \int_{|y| \geq \eta} |\phi_t(y)|^q dt &= \int_{|y| > \eta} t^{-nq} |\phi(y/t)|^q dy \\
 &= t^{n(1-q)} \int_{|z| > \eta/t} |\phi(z)|^q dz \\
 &\leq t^{n(1-q)} \int_{|z| > \eta/t} (1 + |z|)^{-q(n+\epsilon)} dz \\
 &= c_n \int_{\eta/t}^{\infty} r^{n-1} (1+r)^{-q(n+\epsilon)} dr \\
 \text{(bound the integral)} \quad &\leq c_n t^{n(1-q)} \frac{1}{q(n+\epsilon) - n} \left(\frac{n}{t}\right)^{n-q(n+\epsilon)} \\
 &= c_{n,q,\eta} t^{q\epsilon}
 \end{aligned}$$

Therefore $I_q(t) \rightarrow 0$ for $q < \infty$.

Thus what happens outside of the ball goes to 0, so our estimate depends on the dynamics inside the small ball. We will decompose the ball into annular regions, which is a technique used in harmonic analysis for more involved bounds. We are doing this proof as much for the result as we are to illustrate the technique.

Choose t so small such that $\eta/t \geq 1$. Then choose $k_0 \in \mathbb{N}$ such that $2^{k_0} \leq \eta/t < 2^{k_0+1}$. For $2^{-k-1}\eta \leq |y| < 2^{-k}\eta$ for $k = 0, \dots, k_0 - 1$. Then $|\phi_t(y)| \leq c_n t^\epsilon 2^{(k+1)(n+\epsilon)} \eta^{-n-\epsilon}$. We substitute the estimate then change to the annular region.

$$\int_{|y| < \eta} |f(x-y) - f(x)| |\phi_t(y)| dy = \sum_{j=1}^{k_0-1} \int_{\{2^{-j}\eta < |y| < 2^{-j+1}\eta\}} + \int_{\{|y| \leq 2^{-k_0}\eta\}}$$

Now recall that for $j \geq 1$, $2^{-j+1}\eta \leq \eta$. Since x is a Lebesgue point, the first expression is

$$\begin{aligned}
 &\leq C t^\epsilon \eta^{-n-\epsilon} \delta \sum_{j=1}^{k_0-1} 2^{j(n+\epsilon)} (2^{-j+1})^n \eta^n + C t^{-n} (2^{-k_0}\eta)^n \delta \\
 &\leq C t^\epsilon \eta^{-\epsilon} \delta t \sum_{j=0}^{k_0} 2^{j\epsilon} + C t^{-n} \eta^n 2^{-k_0+n} \delta.
 \end{aligned}$$

Thus we get the estimate

$$\int_{|y| < \eta} |f(x-y) - f(x)| |\phi_t(y)| dy \leq C \delta \left\{ \left[\left(\frac{t}{\eta}\right) 2^{k_0+1} \right]^\epsilon + \left(\frac{\eta}{t} 2^{-k_0}\right)^n \right\}.$$

But $t/\eta 2^{k_0} \leq 1$ and $\eta/t 2^{-k_0} \leq 2$; hence we have a constant bound $C\delta$. All in all, we've obtain $|f * \phi_t(x) - af(x)| \leq C\delta + o(t)$, so we're done.

Lecture 25: May 28

Proposition 25.1. C_c^∞ and \mathcal{S} are dense in L^p ($p < \infty$) and in $C_0(\mathbb{R}^n)$.

Proof. We start with $f \in L^p$; for any $\epsilon > 0$, there is a $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p < \epsilon$. Recall the function

$$\Psi(x) = \exp \frac{1}{|x|^2 - 1} \chi_{B_1} \in C_c^\infty.$$

More precisely, $\text{supp } \phi = \overline{B_1}$. Normalizing, we obtain

$$\phi = \frac{\Psi}{\int \Psi} \in C_c^\infty.$$

Then $\int \phi = 1$ and $\phi_t * g \in C_c^\infty$. For sufficiently small t , we obtain

$$\|g * \phi_t - g\|_p < \epsilon.$$

We are taking $\alpha = 1$ in the result from last time.

For the $C_0(\mathbb{R}^n)$ result, it suffices to approximate functions of compact support (by an earlier density result). Hence using the bound $\|g * \phi_t - g\|_u < \epsilon$, we obtain the result. \square

We move on to the C^∞ -Urysohn Lemma. Now we want to do partitions of unity with smooth functions, so that you can still differentiate without wrecking things (useful in PDE).

Lemma 25.2. If $K \subset U \subset \mathbb{R}^n$ with K compact, U open, then

$$\chi_K \leq f \leq \chi_U, \quad f \in C_c^\infty.$$

Proof. Set $\delta = d(K, U^c)/3 > 0$, and keep ϕ as before. Define the set

$$V = \{x \in \mathbb{R}^n : d(x, K) < \delta\}.$$

It follows that the function ϕ_δ satisfies

$$\text{supp } \phi_\delta = \overline{B_\delta}.$$

We set $f = \chi_V * \phi_\delta$, so that

$$f(x) = \int \phi_\delta(y) \chi_V(x - y) dy.$$

If $x \in K$ and $|y| < \delta$, then $x - y \in V$. Hence $f(x) = 1$.

Moreover, if $x \notin U$ then for $|y| < \delta$ we have $x - y \notin V$ then $f(x) = 0$. It is clear that $|f| \leq 1$ (by a simple integral bound), so we have constructed the desired approximation f . \square

Fourier Transform

This is another operation that “smooths things out”, but it also picks up other interesting behavior. We can see the size by convolving, but we can’t see oscillations.

Definition 25.3. Consider $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f is

$$\widehat{f}(\xi) = \int f(x) \exp -2\pi i \langle x, \xi \rangle dx.$$

\widehat{f} is nicer than f :

1. $\|\widehat{f}\|_{\infty} \leq \|f\|_1$
2. \widehat{f} is continuous, by dominated convergence:

$$|\widehat{f}(\xi_1) - \widehat{f}(\xi_2)| = \left| \int f(x) (\exp -2\pi i \langle \xi_1, x \rangle - \exp -2\pi i \langle \xi_2, x \rangle) dx \right| \rightarrow 0,$$

since the integrand is dominated by $2|f(x)|$.

Consider the operator

$$\widehat{\cdot}: L^1(\mathbb{R}^n) \rightarrow BC(\mathbb{R}^n).$$

Theorem 25.4. For $f, g \in L^1(\mathbb{R}^n)$,

(i) $\widehat{\tau_y f}(\xi) = \widehat{f}(\xi) \exp -2\pi i \langle \xi, y \rangle$ and $\tau_\eta \widehat{f}(\xi) = \widehat{h}$ where

$$h(x) = f(x) \exp 2\pi i \langle x, \eta \rangle.$$

(ii) If $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an invertible linear transformation, set $S = (T^{-1})^*$. Then

$$\widehat{f \circ T} = |(\det T)^{-1}| \widehat{f} \circ S.$$

In particular, for rotations T we have $\widehat{f \circ T} = \widehat{f} \circ T$, and taking $Tx = t^{-1}x$ for $t > 0$ yields $\widehat{f}_t(\xi) = \widehat{f}(t\xi)$.

(iii) $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$

(iv) If $x^\alpha f \in L^1$ for $|\alpha| \leq k$, then $\widehat{f} \in C^k$ and

$$\partial_\alpha \widehat{f} = [(-2\pi i x)^\alpha f]^\wedge$$

(v) If $f \in C^k$ and $\partial_\alpha f \in L^1$ for $|\alpha| \leq k$ and also $\partial_\alpha f \in C_0$ for all $|\alpha| \leq k - 1$, then

$$\widehat{\partial_\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi).$$

(vi) The Riemann-Lebesgue lemma says

$$\widehat{\cdot}: L^1(\mathbb{R}^n) \rightarrow C_0(\mathbb{R}^n).$$

Proof. Part (i) follows immediately. We’ll do the rest next time. □

Lecture 26: May 30

Properties

Consider $f, g \in L^1(\mathbb{R}^n)$.

2) For $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ linear and invertible, set $S = (T^{-1})^*$. Then

$$\begin{aligned} \widehat{f \circ T}(\xi) &= \int f(Tx) \exp -2\pi i \langle \xi, x \rangle dx \\ (z = Tx) &= |\det T|^{-1} \int f(z) \exp -2\pi i \langle \xi, T^{-1}z \rangle dz \\ &= |\det T|^{-1} \widehat{f}(S\xi), \end{aligned}$$

since $\langle \xi, T^{-1}z \rangle = \langle S\xi, z \rangle$.

3) $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$, since

$$\begin{aligned} \widehat{f * g}(\xi) &= \int f * g \exp -2\pi i \langle \xi, x \rangle dx \\ &= \int \int f(x - y)g(y) \exp -2\pi i \langle \xi, x \rangle dy dx \\ (\text{Fubini}) &= \int g(y) \left(\int f(x - y) \exp -2\pi i \langle \xi, x - y \rangle dx \right) \exp -2\pi i \langle \xi, y \rangle dy \\ &= \widehat{f}(\xi) \widehat{g}(\xi). \end{aligned}$$

4) We differentiate with respect to ξ (which is what the subscript means). Using induction on $|\alpha|$, L^1 -type bounds on all the derivative of $\exp -2\pi i \langle \xi, x \rangle$, and dominated convergence, it follows that

$$\begin{aligned} \partial_\xi^\alpha \widehat{f}(\xi) &= \partial_\xi^\alpha \int f(x) \exp -2\pi i \langle \xi, x \rangle dx \\ &= \int f(x) \partial_\xi^\alpha \exp -2\pi i \langle \xi, x \rangle dx \\ &= \int f(x) (-2\pi i x)^\alpha \exp -2\pi i \langle \xi, x \rangle dx \\ &= [-(2\pi i x)^\alpha f]^\wedge(\xi). \end{aligned}$$

5) If $f \in C^k$, $\partial^\alpha f \in L^1$, $|\alpha| \leq k$. Suppose that $\partial^\alpha f \in C_0$ whenever $|\alpha| \leq k - 1$. Then

$$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi).$$

Now if $|\alpha| = 1$, for $f \in C_0 \cap C^1$ we have

$$\begin{aligned} \int [\partial_j f(x)] \exp -2\pi i \langle x, \xi \rangle dx &= \int \partial_j (f(x) \exp -2\pi i \langle x, \xi \rangle) dx \\ &\quad - \int f(x) \partial_j \exp -2\pi i \langle x, \xi \rangle dx \\ &= 0 - \int f(x) (-2\pi i \xi_j) \exp -2\pi i \langle x, \xi \rangle dx \\ &= 2\pi i \xi_j \widehat{f}(\xi). \end{aligned}$$

We are using that $f(x) \exp -2\pi i \langle x, \xi \rangle \in C_0$, so the integral decays. Specifically, if f is of compact support we are done. Now we use density to extend. On each ball, you look at $f - f_n$.

By induction, we can assume it holds for $|\alpha| = k$. So for any α of length $k + 1$, we can write $\alpha = \alpha' + \beta$ for $|\beta| = 1$. Now since $\partial^\alpha f = \partial^\beta (\partial^{\alpha'} f)$, the result follows.

The Fourier transform sends L^1 into BC; but the Riemann-Lebesgue lemma gives us something a little bit better; we actually end up in $C_0(\mathbb{R}^n)$. To prove this, we use that C_c^∞ (smooth functions of compact support) is dense in $L^1(\mathbb{R}^n)$. Observe that

$$|\widehat{\partial_j f}(\xi)| = 2\pi |\xi_j| |\widehat{f}(\xi)| \leq M.$$

Hence $\widehat{f} \in C_0(\mathbb{R}^n)$. For $f \in L^1$, take a sequence $f_n \rightarrow f$ with $f_n \in C_c^\infty$. We've see that

$$\|\widehat{f} - \widehat{f}_n\|_u \leq \|f - f_n\|_1,$$

so $\widehat{f}_n \rightarrow \widehat{f}$ in the uniform norm. But $C_0(\mathbb{R}^n)$ is closed in the uniform norm, so $\widehat{f} \in C_0(\mathbb{R}^n)$.

Corollary 26.1. $\widehat{\cdot}: \mathcal{S} \rightarrow \mathcal{S}$, and this map is "continuous" in the right sense.

Proof. Consider $f \in \mathcal{S}$. Then $x^\alpha \partial^\beta f \in L^1 \cap C_0$ for every α, β . Now consider

$$\begin{aligned} L^\infty \ni \widehat{x^\alpha \partial^\beta f}(\xi) &= (-2\pi i)^{|\alpha|} \partial^\alpha (\widehat{\partial^\beta f}(\xi)) \\ &= (-2\pi i)^{|\alpha|} (2\pi i)^{|\beta|} \partial^\alpha (\xi^\beta \widehat{f}(\xi)). \end{aligned}$$

Hence the latter is in L^∞ . By our characterization of \mathcal{S} , it follows that $\widehat{f} \in \mathcal{S}$.

Now we show that the Fourier transform acts continuously in the sense of a Frechet space. It suffices to show that given a sequence $f_m \in \mathcal{S}$ such that

$$\|f_m - f\|_{(N, \alpha)} \rightarrow 0 \text{ for every } N, \alpha,$$

we have $\|\widehat{f}_m - \widehat{f}\|_{(N, \alpha)} \rightarrow 0$ for each N, α as well.

In order to get this type of estimate, it will suffice to prove some relationships among seminorms, which we then apply to $f = f_m - f$ (i.e., the following f is not the same f as above).

$$\begin{aligned} \|\widehat{x^\alpha \partial^\beta f}\|_u &\leq \|x^\alpha \partial^\beta f\|_{L^1} \\ &\leq C_n \|(1 + |x|)^{n+1} x^\alpha \partial^\beta f\|_u \\ &\leq C_n \|f\|_{(n+1+\alpha, \beta)}. \end{aligned}$$

Hence we have the bound

$$\|\partial^\alpha(\xi^\beta \hat{f})\|_{\mathbf{u}} \leq C_n \|f\|_{(n+1+|\alpha|, \beta)}.$$

If $|\alpha| = 0$, we are done. Now for $|\alpha| = 1$, we have

$$\partial^\alpha(\xi^\beta \hat{f}) = (\delta^\alpha \xi^\beta) \hat{f} + \xi^\beta \partial^\alpha \hat{f}.$$

Then we obtain

$$\begin{aligned} \|(1 + |\xi|)^{|\beta|} \partial^\alpha f\|_{\mathbf{u}} &\leq \|\partial^\alpha(\xi^\beta f)\|_{\mathbf{u}} + \|\xi^{\beta-\alpha} \hat{f}\|_{\mathbf{u}} + \|\partial^\alpha \hat{f}\|_{\mathbf{u}} \\ &\leq C_n \left[\|f\|_{(n+1+|\alpha|, \beta)} + \|f\|_{(n+1, \beta-\alpha)} + \|f\|_{(n+1+|\alpha|, 0)} \right], \end{aligned}$$

and something similar holds for $|\alpha| > 1$.

Then we eventually obtain

$$\|\hat{f}\|_{(N, \alpha)} \leq C_{N, \alpha} \sum_{|\gamma| \leq N+|\alpha|} \|f\|_{(N+n+1, \gamma)},$$

and the original continuity statement follows. \square

The significance of this is that we can prove statement for Schwarz functions, which are then dense in “everything”.

Lecture 27: June 2

Proposition 27.1. Let $f(x) = \exp -\pi a|x|^2$ for $a > 0$ and $x \in \mathbb{R}^n$. Then

$$\hat{f}(\xi) = a^{-n/2} \exp -\pi|\xi|^2/a.$$

Proof. We do $n = 1$ then extend via Fubini. So $f(x) = \exp -\pi ax^2$ and $f'(x) = -2\pi ax \exp -\pi ax^2$. Hence

$$\begin{aligned} (\hat{f}(\xi))' &= [-2\pi i x f]^\wedge(\xi) \\ &= \frac{i}{a} \hat{f}'(\xi) \\ &= 2\pi a \xi \hat{f}(\xi). \end{aligned}$$

Consequently

$$\begin{aligned} \hat{f}'(\xi) &= \frac{-2\pi a}{\xi} \hat{f}(\xi) \\ \implies 0 &= \frac{d}{d\xi} \left(\hat{f}(\xi) \exp \frac{\pi}{a} \xi^2 \right) \\ \implies a^{-1/2} &= \hat{f}(0) = \hat{f}(\xi) \exp \frac{\pi}{a} \xi^2. \end{aligned}$$

Thus the claim is proven for $n = 1$.

For $n > 1$, observe that

$$\begin{aligned} \hat{f}(\xi) &= \int \exp -\pi a \sum_{j=1}^n x_j^2 - 2\pi i \sum_{j=1}^n \xi_j x_j \, dx_1 \cdots dx_n \\ &= \prod_{j=1}^n \left(\int \exp -\pi a x_j^2 - 2\pi i \xi_j x_j \, dx_j \right) \\ &= \prod_{j=1}^n \left(a^{-1/2} \exp -\frac{\pi}{a} \xi_j^2 \right) \\ &= a^{-n/2} \exp -\frac{\pi}{a} |\xi|^2. \end{aligned}$$

□

Theorem 27.2 (Fourier inversion formula). If $f \in L^1$ and $\hat{f} \in L^1$, then f agrees a.e. with a continuous function f_0 such that

$$\check{f} = [\hat{f}]^\vee = f_0.$$

where $\check{f} = \int f(\xi) \exp 2\pi i \langle \xi, x \rangle d\xi$.

Lemma 27.3. For $f, g \in L^1$, we have

$$\int f \hat{g} = \int \hat{f} g.$$

Proof. It's just Fubini:

$$\begin{aligned} \int f(\xi) \hat{g}(\xi) \, d\xi &= \int f(\xi) \left(\int g(x) \exp -2\pi i \langle x, \xi \rangle \right) \, d\xi \\ \text{(Fubini)} \quad &= \int g(x) \left(\int f(\xi) \exp -2\pi i \langle x, \xi \rangle \, d\xi \right) \, dx \\ &= \int g \hat{f}. \end{aligned}$$

□

Proof of Fourier inversion. Fix $t > 0$ and $x \in \mathbb{R}^n$. Then define

$$\phi(\xi) = \exp 2\pi i \langle x, \xi \rangle \exp -\pi t^2 |\xi|^2 \in \mathcal{S}.$$

It follows that

$$\begin{aligned} \hat{\phi}(y) &= \tau_x \left(\widehat{\exp -\pi t^2 |\xi|^2} \right) (y) \\ &= \tau_x \left(t^{-n} \exp -\pi \frac{|y|^2}{t^2} \right) \\ &= t^{-n} \exp -\pi \frac{|x - y|^2}{t^2}. \end{aligned}$$

Meanwhile, we have

$$\begin{aligned} \hat{\phi}(y) &= \int \exp 2\pi i \langle x, \xi \rangle \exp -\pi t^2 |\xi|^2 \exp -w\pi i \langle \xi, y \rangle \, d\xi \\ &= \int \exp 2\pi i \langle xi, x - y \rangle \exp -\pi t^2 |\xi|^2 \, d\xi \\ &= t^{-n} \exp -\pi \frac{|x - y|^2}{t^2}. \end{aligned}$$

We make the definition $g_t(x) = t^{-n} \exp -\pi \frac{|x|^2}{t^2}$, so the last expression is just $g_t(x - y)$. Also set $g(x) = \exp -\pi |x|^2$. We then have

$$\begin{aligned} \int \phi(\xi) \hat{f}(\xi) \, d\xi &= \int \exp -\pi t^2 |\xi|^2 \exp 2\pi i \langle \xi, x \rangle \hat{f}(\xi) \, d\xi \\ \text{LDCT} \quad &\rightarrow [\hat{f}]^\vee(x). \end{aligned}$$

Simultaneously, observe that

$$\begin{aligned} \hat{\phi}(\xi) f(\xi) \, d\xi &= \int f(y) g_t(x - y) \, dy \\ &= f * g_t(x) \\ &\rightarrow f, \end{aligned}$$

since $\int f(x) dx = 1$. Indeed, there is a sequence $t_i \rightarrow 0$ such that $f * g_{t_i} \rightarrow f$ pointwise a.e. in x . Thus f agrees a.e. with $[\hat{f}]^\vee$, and a symmetric argument holds for $\widehat{\check{f}}$. Hence we obtain the result. \square

Corollary 27.4. For $f \in L^1$, if $\hat{f} = 0$ then $f = 0$ a.e.

Corollary 27.5. $\hat{\cdot}: \mathcal{S} \rightarrow \mathcal{S}$ is an isomorphism.

For $f \in \mathcal{S}$ we have $\check{f}(x) = \hat{f}(-x)$. Hence we obtain a correspondence between the norms of f and \hat{f} .

Plancherel Theorem

This is for the homework; specifically, problem 2 of homework 9.

If $f \in L^1 \cap L^2$ then $\hat{f} \in L^2$ and $\hat{\cdot}: L^1 \cap L^2 \rightarrow L^2$ extends uniquely to an isometry of L^2 (i.e. $\|\hat{f}\|_2 = \|f\|_2$).

Proof of Plancherel. Let $X = \{f \in L^1: \hat{f} \in L^1\}$. Notice that $\mathcal{S} \subset X$. Now if $f \in X$, we have $\hat{f} \in L^1$. Now by Fourier Inversion, we have $f \in L^\infty$. It follows that

$$\int |f|^2 dx \leq \|f\|_\infty \int |f| dx < \infty \implies f \in L^2.$$

Note that this is an interpolation result: $L^1 \cap L^\infty \subset L^2$.

It follows that for $f \in X$, we have $f \in L^2$. Since the Schwarz functions are dense in L^2 , it follows that X is dense in L^2 .

Claim 27.6. For $f, g \in X$, we have $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$.

Proof. Let $h = \overline{\hat{g}}$. Using Fourier Inverse, we have $\hat{h}(\xi) = \overline{g(\xi)}$. Therefore

$$\begin{aligned} \int f \overline{g} &= \int f \hat{h} \\ &= \int \hat{f} h \\ &= \int \hat{f} \overline{\hat{g}}. \end{aligned}$$

Thus $f = g$, so $\|\hat{f}\|_2 = \|f\|_2$. We will finish the proof next time. \square

\square

\square

Lecture 28: June 4

Theorem 28.1 (Plancherel). *If $f \in L^1 \cap L^2$ then $\hat{f} \in L^2$ and $\wedge: L^1 \cap L^2 \rightarrow L^2$ is an isometry.*

Proof. We defined the subset

$$X = \{f \in L^1: \hat{f} \in L^2\} \subset L^1 \cap L^2,$$

and showed that $\|\hat{f}\|_2 = \|f\|_2$ for $f \in X$.

Now we extend \wedge continuously to L^2 . If $f \in L^2$, pick a sequence $f_n \rightarrow f$ with $f_n \in X$. Now define an operator T by setting $Tf = \lim_{n \rightarrow \infty} \hat{f}_n$.

Claim 28.2. (1) T is well-defined

(2) $Tf = \hat{f}$ for $f \in L^1 \cap L^2$

Proof. (1) Observe that $\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2$. Thus the sequence $\{\hat{f}_n\}$ is Cauchy in L^2 , so it converges. The definition doesn't depend on the choice of convergent sequence $\{f_n\}$, since if $\{g_n\}$ is any other sequence converging to f we have

$$\begin{aligned} \|\hat{g}_n - \hat{f}_n\|_2 &= \|g_n - f_n\|_2 \\ &\leq \|g_n - f\|_2 + \|f - f_n\|_2. \end{aligned}$$

Hence \hat{g}_n and \hat{f}_n have the same limit, so Tf is independent of the choice of convergent sequence.

(2) Take $g(x) = e^{-\pi|x|^2}$, and set $g_t(x) = t^{-n}e^{-\pi|x|^2/t^2}$. For $f \in L^1 \cap L^2$, it follows that $f * g_t$ is Schwarz, and hence L^1 . Therefore

$$\begin{aligned} \widehat{f * g_t}(\xi) &= \hat{f}(\xi)\hat{g}_t(\xi) \\ &= e^{-\pi|\xi|^2 t^2} \hat{f}(\xi). \end{aligned}$$

Now since $f \in L^1$, we have that \hat{f} is bounded and therefore $f * g_t \in X$. Observe that $\int g = 1$. Thus we have

$$f * g_t \rightarrow f, \quad t \rightarrow 0$$

where the convergence holds in both L^1 and L^2 . Now since

$$\|\widehat{f * g_t} - \widehat{f * g_s}\|_2 = \|f * g_t - f * g_s\|_2,$$

it follows that $\widehat{f * g_t}$ also converges in L^2 .

Next, observe that

$$\widehat{f * g_t}(\xi) = e^{-\pi|\xi|^2 t^2} \hat{f}(\xi) \rightarrow \hat{f}$$

where the convergence is uniform on compact sets (but pointwise would have sufficed). Now $f * \widehat{g}_t$ has a subsequence that converges to Tf pointwise a.e., so by uniqueness of limits we have $\widehat{f * g_t} \rightarrow Tf$ in L^2 .

Lastly, observe that

$$\begin{aligned} \|Tf\|_2 &= \|\widehat{f}\|_2 \\ &= \lim_{t \rightarrow 0} \|\widehat{f * g_t}\|_2 \\ &= \lim_{t \rightarrow 0} \|f * g_t\|_2 \\ &= \|f\|_2. \end{aligned}$$

□

□

The rest of today is purely formal (assume that all limiting operations go through).

Application to PDE

Definition 28.3. A **linear differential operator** is an operator of the form

$$Lf = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha f(x).$$

If a_α is constant in x for all α , we say that L has **constant coefficients**.

Our goal is to formally solve the equation $Lu = f$ on \mathbb{R}^n , where L has constant coefficients. In order to do so, we compute $\widehat{Lu} = \widehat{f}$. This yields

$$\sum_{|\alpha| \leq m} a_\alpha \partial^\alpha u(\xi) = \sum_{|\alpha| \leq m} a_\alpha (2\pi i \xi)^\alpha \widehat{u}(\xi).$$

Definition 28.4. The **symbol** of L is defined to be

$$P(\xi) = \sum_{|\alpha| \leq m} a_\alpha (2\pi i \xi)^\alpha.$$

Therefore $P(\xi)\widehat{u}(\xi) = \widehat{f}(\xi)$. Hence if we can find ϕ such that $P^{-1} = \widehat{\phi}$, we will obtain

$$u = (\widehat{\phi \widehat{f}})^\vee = \phi * f.$$

Recall the PDE problem from homework 8; we were solving the Laplacian $\Delta u = 0$ on \mathbb{R}_+^{n+1} with boundary condition $u = f$ on the boundary. We needed $f \in BC(\mathbb{R}^n \times \{0\})$. Using this new technique, we can solve the differential equation for more general f . Specifically, we allow $f \in L^p(\mathbb{R}^n \times 0)$ for $p \in [1, \infty]$. The boundary condition $u = f$ becomes more complicated, since we need to control L^p norms now.

Consider the Laplacian Δ on \mathbb{R}^n . We compute its symbol

$$P(\xi) = \sum_{j=1}^n (2\pi i)^2 \xi_j^2 = -4\pi^2 |\xi|^2.$$

Now we write

$$\Delta_{\mathbb{R}_+^{n+1}} = \Delta_{\mathbb{R}^n} + \frac{\partial^2}{\partial t^2},$$

where we identify \mathbb{R}_+^{n+1} with (x, t) for $t > 0$ and $x \in \mathbb{R}^n$.

Taking Fourier transforms, we find that $\Delta_{\mathbb{R}_+^{n+1}} u = 0$ and $u(x, 0) = f(x)$ is equivalent to solving

$$-4\pi^2 |\xi|^2 \hat{u}(\xi, t) + \frac{\partial^2}{\partial t^2} \hat{u}(\xi, t) = 0, \quad \hat{u}(\xi, 0) = \hat{f}(\xi).$$

Using regularity at infinity (i.e. dropping terms that blow up), we have $\hat{u}(\xi, t) = C(\xi) e^{-2\pi t |\xi|}$. Then we obtain the solution

$$\hat{u}(\xi, t) = \hat{f}(\xi) e^{-2\pi t |\xi|}.$$

We obtain the **Poisson kernel** of \mathbb{R}_+^{n+1} , given by

$$P_t = [e^{-2\pi t |\xi|}]^\wedge = c_n \frac{t}{(t^2 + |x|^2)^{(n+1)/2}}.$$

Therefore we obtain

$$\hat{u}(\xi, t) = \hat{P}_t(\xi) \hat{f}(\xi) = \widehat{P_t * f}(\xi),$$

which means $u(x, t) = P_t * f(x)$. We obtain $\|P_t * f - f\|_{L^p(\mathbb{R}^n \times \{0\})} \rightarrow 0$ as $t \rightarrow 0$.

Math 526 Real Analysis

Notes by [Avi Levy](#)

Lecture 29: June 6

1. Measure and integration (Q1)
2. Differentiation (Q1/Q2)
3. Topology (and metric spaces) (think of in class only - no need \mathbb{T} , metrizable)
4. Functional analysis (a little L^p at end of Q2, Hilbert spaces, compact operators)
5. L^p spaces
6. Extra topics: Radon measures, Fourier analysis

Measure and integration

1. Convergence theorems (Fatou, LDCT, Monotone, modes of convergence, simple functions used to approximate everything)
2. Egoroff (finite measure only!)
3. Fubini

Differentiation

1. Lebesgue differentiation theorem
2. Radon-Nikodym
3. BV, AC, and FTC
4. covering lemma

Topology and metric spaces

1. Arzela-Ascoli
2. Stone-Weierstrass
3. Urysohn's Lemma
4. Baire Category
5. No Tychonoff

Functional analysis

1. Hahn-Banach (especially theorem 5.8 that's a consequence; more useful)
2. OMT/Closed graph/UBP (all from Baire)
3. Hilbert spaces (especially bases, closest points)
4. Riesz Rep (In L^p , for \mathbb{R}^n , for Radon measures, Hilbert spaces; her very favorite)
5. Compact operators (spectral theorem and so on)

L^p spaces

1. Duality (also Riesz Rep)
2. Inequalities: Hölder, Minkowski (integral version).

Addition topics

1. Dual of L^∞ ; we only really did C_0 on compact sets. Then we exhausted to work on $C_c(\mathbb{R}^n)$. Folland has lots of extra things we didn't do.
2. Fourier transform; the key points were Plancherel and IFT (the key things from the homework problem)
3. Vitali

The Vitali you SHOULD use:

You're in a separable metric space X , and let \mathcal{F} be a collection of non-degenerate closed balls such that sup of diameters of the balls is finite. Then there exists $\{B_i\}$, a disjoint countable subcollection of \mathcal{F} , such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{i=1}^{\infty} 5B_i.$$

Moreover, if $B \in \mathcal{F}$, there is an i such that $B \cap B_i$ is non-empty and $B \subset 5B_i$.

There were some useful consequences. For instance, in \mathbb{R}^n and you had an open set, then you could cover the set in measure with a countable collection of disjoint balls. We might need to work with doubling measures for Vitali, but certainly not a Radon measure (like in GMT).

Reminder of convergence

Claim 29.1 (Prelim 2000 #6).

$$\int_0^1 \left(\sum_{k=1}^{\infty} \frac{x^k \cos(2^k \pi x)}{k} \right) dx = \sum_{k=1}^{\infty} \int_0^1 \frac{x^k \cos(2^k \pi x)}{k} dx.$$

Proof. The Harish approach is always Fubini, so we start trying this out (where dv is counting measure and dx is normal). By Fubini, $f_k(X) \in L^1[0, 1]$ for every k .

More straightforward to just use dominated. Then set $f_n(x)$ to be the partial sum. For $x \in [0, 1)$, we have

$$0 \leq |f_n(x)| \leq \sum_{k=1}^n x^k/k = g_n(x).$$

By general dominated convergence, if $\int g_n \rightarrow \int g$ for some $g \in L^1$, and $f_n \rightarrow f$ a.e., then

$$\int f_n \rightarrow \int f.$$

Note that we only know f on $[0, 1)$, it might not exist at $x = 1$.

The last detail to check was that $g \in L^1$ and $\int g_n \rightarrow \int g$. By MCT, it follows that $g \in L^1$ (and we could have done this without generalized LDCT). The reason for the cosine was to force us to use MCT; any bounded measurable would work in its place (for the proof to go through). Don't need to show measurable before integrating. \square

Read the exam in the first 5 minutes, classify the problems by what theorem they will require (what's the key). After you've used the 5 minutes to do that, then start with the easiest one. This is a point competition, not to impress her about doing the most difficult one. There are 8 problems, 4 hours it's a prelim.

Let f be $AC[0, 1]$, $p \in (1, \infty)$ assume f' in L^p prove

$$\frac{|f(x) - f(a)|}{|x - a|^{1/q}} \quad x \rightarrow a.$$

We are using absolute continuity of integration, FTC, and Hölder.

$$\frac{|f(x) - f(a)|}{|x - a|^{1/q}} \leq \left(\int_a^x |f'|^p(t) dt \right)^{1/p}$$

then take limit as $x \rightarrow a$ to finish.